PHYSICS 200B : CLASSICAL MECHANICS
FINAL EXAMINATION WINTER 2016

(1) Consider the nonlinear oscillator described by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2 + \tfrac{1}{4}\epsilon a q^4 + \tfrac{1}{4}\epsilon b p^4 \quad ,$$

where ε is small.

(a) Find the perturbed frequencies $\nu(J)$ to lowest nontrivial order in ϵ.

(b) Find the perturbed frequencies $\nu(A)$ to lowest nontrivial order in ϵ, where A is the amplitude of the q motion.

(c) Find the relationships $\phi = \phi(\phi_0, J_0)$ and $J = J(\phi_0, J_0)$ to lowest nontrivial order in ϵ.

(2) Consider the forced modified van der Pol equation,

$$\ddot{x} + \epsilon(x^4 - 1)\,\dot{x} + x = \epsilon f_0 \cos(t + \epsilon\nu t) \quad ,$$

where ϵ is small. Carry out the multiple scale analysis to order ϵ. Following §3.3.2 in the Lecture Notes, find and analyze the equation which relates the amplitude A, detuning ν, and force amplitude f_0 for entrained oscillations. Perform the requisite linear stability analysis and make a plot similar to that in Fig. 3.4 of the Lecture Notes. Is there a region of entrained oscillations which exhibits hysteresis as the detuning parameter is varied? If so, find the corresponding range of f_0 over which this occurs.

Bonus: Use `Mathematica` *or* `Matlab` *to integrate the equation, showing examples of entrained and heterodyne behavior, as in Fig. 3.6 (1000 Quatloos extra credit).*

(3) Consider shock formation in the inviscid Burgers' equation, $c_t + c\,c_x = 0$. Let the function $c(\zeta) = c(x = \zeta, t = 0)$ be given by the triangular profile,

$$c(\zeta) = c_0 \left(\frac{a - |\zeta|}{a}\right) \Theta\big(a - |\zeta|\big) \quad .$$

(a) Find the break time t_{B}.

(b) Implement the shock fitting protocol and find $\zeta_-(t)$, $\zeta_+(t)$, and $x_{\mathrm{s}}(t)$.

(c) Find the shock discontinuity $\Delta c(t)$ for $t > t_{\mathrm{B}}$.

(d) Sketch $c(x,t)$ *vs.* x for $t/t_{\mathrm{B}} = 0$, $\frac{1}{2}$, 1, 2, and 4. Show that for $t \geq t_{\mathrm{B}}$, $c(x,t)$ *vs.* x has the form of a right triangle whose area is given by $\int\limits_{-a}^{a} d\zeta\, c(\zeta)$.

(e) Without shock fitting, sketch the characteristics in the (x,t) plane and highlight the region where they cross. Then sketch the characteristics after shock fitting. *Hint: Your sketches should roughly resemble those in Fig. 4.13 of the Lecture Notes.*