Physics 4E **Quiz 3 Solutions** Spring 2017

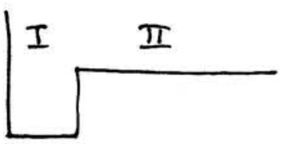
Problem 1

(a) For ∞ well, $E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$; for $n=1$, $L = \frac{3}{4}\pi$ Å,

$$E_1 = \frac{\hbar^2\cancel{\pi^2}\cdot\cancel{16}^{\,8}}{\cancel{2}m\cdot 9\cdot\cancel{\pi^2}\,Å^2} = \frac{8}{9}\frac{\hbar^2}{m\cdot Å^2} = \frac{8}{9}\times 7.62\,eV \Rightarrow \boxed{E_1 = 6.77\,eV}$$

(b) $E_1 = 3.81\,eV \Rightarrow V_0 < \infty$

From $E_1 = \frac{\hbar^2k^2}{2m_e} \Rightarrow 3.81 = \frac{7.62}{2}k^2Å^2 \Rightarrow \boxed{k = 1\,Å^{-1}}$



$\psi_I = A\sin(kx)$ continuity: $A\sin(kL) = B$ $\Rightarrow$

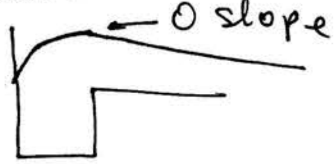
$\psi_{II} = Be^{-\alpha(x-L)}$ $\Rightarrow$ cont. of ψ': $kA\cos(kL) = -\alpha B$

$\Rightarrow \boxed{\frac{\cos(kL)}{\sin(kL)}\cdot k = -\alpha}$. For $k = 1\,Å^{-1}$, $L = \frac{3}{4}\pi$ Å $\Rightarrow kL = \frac{3}{4}\pi$

$\Rightarrow \cos\left(\frac{3}{4}\pi\right) = -\frac{\sqrt{2}}{2}$, $\sin\left(\frac{3}{4}\pi\right) = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\alpha = k = 1\,Å^{-1}}$

$\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \Rightarrow V_0 - E = 3.81\,eV \Rightarrow \boxed{V_0 = 7.62\,eV}$

(d) Minimum V_0 that will bind an electron gives a wavefunction:



$\Rightarrow kL = \frac{\pi}{2} \Rightarrow k = \frac{\cancel{\pi}\cdot\cancel{4}^{\,2}}{\cancel{2}\cdot 3\cancel{\pi}\,Å} = \frac{2}{3}\,Å^{-1}$

$E_1 = \frac{\hbar^2k^2}{2m_e} = \frac{\hbar^2}{m_e}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{4}^{\,2}}{9}Å^{-2} = \frac{2}{9}\times 7.62\,eV = 1.693\,eV$

So minimum V_0 is $\boxed{V_0 = 1.693\,eV}$ $(V_0 = E_1)$

Problem 2

(a) Normalization:

$$1 = \int_{-\infty}^{\infty} dx\, |\Psi(x)|^2 = \int_{-\infty}^{\infty} dx\ C^2 e^{-\alpha x^2} = C^2 \sqrt{\frac{\pi}{\alpha}} \Rightarrow$$

$$\Rightarrow C^2 = \sqrt{\frac{\alpha}{\pi}} \Rightarrow \boxed{C = \left(\frac{\alpha}{\pi}\right)^{1/4}}$$

(b) $p_{op} = \frac{\hbar}{i}\frac{d}{dx}$ is the momentum operator.

$$p_{op}\Psi(x) = -\frac{\hbar}{i}\cdot\frac{\alpha}{2}\cdot 2x \cdot C e^{-\alpha x^2/2} = -\frac{\hbar}{i}\alpha x e^{-\alpha x^2/2}\cdot C$$

$$\Rightarrow \langle p \rangle = \int dx\, \Psi^*(x)\, p_{op}\Psi(x) = -\frac{\hbar}{i}\alpha C^2 \int_{-\infty}^{\infty} dx \cdot x \cdot e^{-\alpha x^2} = 0$$

$\Rightarrow$ $\boxed{\langle p \rangle = 0 \text{ because integral of an odd function is } 0 \text{ (from } -\infty \text{ to } \infty)}$

(c) $p_{op}^2 = -\hbar^2 \frac{d^2}{dx^2} = p_{op}\cdot p_{op}$

$$p_{op}^2 \Psi(x) = \hbar^2 \frac{d}{dx}\left(\alpha x e^{-\alpha x^2/2}\cdot C\right) = C\cdot\hbar^2\left(\alpha - \alpha^2 x^2\right) e^{-\alpha x^2/2}$$

$$\langle p^2 \rangle = \int dx\, \Psi^*(x)\, p_{op}^2\Psi(x) = C^2\hbar^2\cdot\alpha\left[\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} - \alpha\int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha x^2}\right] =$$

$$= C^2\hbar^2\alpha\left[\sqrt{\frac{\pi}{\alpha}} - \alpha\cdot\frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}\right] = \cancel{\sqrt{\frac{\alpha}{\pi}}}\cdot\hbar^2\cdot\alpha\cdot\frac{1}{2}\cancel{\sqrt{\frac{\pi}{\alpha}}}$$

$$\Rightarrow \boxed{\langle p^2 \rangle = \hbar^2\cdot\frac{\alpha}{2}} \quad ; \quad \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \Rightarrow$$

$$\boxed{\Delta p = \left(\frac{\alpha}{2}\right)^{1/2}\hbar}$$

Problem 3

Transmission coefficient

$$T = e^{-2\sqrt{\frac{2m}{\hbar^2}(V_0 - E)}\,a}$$

In the case on the left, $T_1 = \frac{100}{10{,}000} = 0.01 = e^{-2\sqrt{\frac{2m}{\hbar^2}}\cdot\sqrt{\frac{2}{3}V_0}\,a}$

In the case in the middle, $V_0 - E = \frac{1}{3}V_0 \Rightarrow$

$$T_2 = e^{-2\sqrt{\frac{2m}{\hbar^2}}\sqrt{\frac{1}{3}V_0}\,a} = T_1^{\frac{1}{\sqrt{2}}} = 0.01^{\frac{1}{\sqrt{2}}} = 0.0385$$

$\Rightarrow$ $\boxed{\text{385 particles tunnel through.}}$ (a)

(b) $T_3 = e^{-2\sqrt{\frac{2m}{\hbar^2}}\sqrt{\frac{1}{3}V_0}\cdot b} = T_1^{\frac{b}{\sqrt{2}a}} = T_1 \Rightarrow$

$\Rightarrow$ $\boxed{b = \sqrt{2}\,a}\boxed{= 1.41\,a}$

(c) $T_4 = e^{-2\sqrt{\frac{2M}{\hbar^2}}\sqrt{\frac{1}{3}V_0}\,a} = T_1^{\sqrt{\frac{M}{m}}\cdot\frac{1}{\sqrt{2}}} = T_1 \Rightarrow$

$\Rightarrow \sqrt{\frac{M}{m\cdot 2}} = 1 \Rightarrow$ $\boxed{M = 2m}$