Physics 4E — Final Exam — Spring 2017

$$\Delta E = E_0 Z^2\left(1-\frac{1}{n^2}\right) = \frac{hc}{\lambda} \quad ; \; E_0 = 13.6\,eV$$

for transitions from 1 to n

So $\Delta E = \frac{3}{4} E_0 Z^2$ for $1 \to 2$, $\Delta E = E_0 Z^2$ for $1 \to \infty$

$Z=1$: $\lambda = 1216\,Å$, $\lambda = 911.8\,Å$

$Z=2$: $\lambda = 303.9\,Å$, $\lambda = 227.9\,Å$

$Z=3$: $\lambda = 135.1\,Å$, $\lambda = 101.3\,Å$

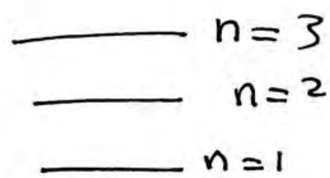
$Z=4$: $\lambda = 76.0\,Å$, $\lambda = 57.0\,Å$

$Z=5$: $\lambda = 48.6\,Å$, $\lambda = 36.5\,Å$

Since $\lambda = 41.03 \Rightarrow Z=5$

For $1 \to 3$ transition, $\Delta E = 25 E_0 \cdot \frac{8}{9} = \frac{hc}{\lambda} \Rightarrow$

$\lambda = 41.03\,Å$



We can have:

n=3 to m=1 : $\lambda = 41.03\,Å$

n=3 to m=2 : $\lambda = 262.6\,Å$

n=2 to m=1 : $\lambda = 48.6\,Å$

(b) For $n=2$, $g=4$; for $n=1$, $g=1 \Rightarrow 4e^{-E_2/kT} = e^{-E_1/kT}$

$\Rightarrow 4 = e^{(E_2-E_1)/kT} \Rightarrow T = \frac{E_2-E_1}{k_B \ln 4} = \boxed{2,133,746\,K}$

$E_2 - E_1 = E_0 Z^2 \cdot \frac{3}{4} = 255\,eV$

Problem 2

$$u(\lambda) = \frac{8\pi}{\lambda^4} \frac{hc/\lambda}{e^{hc/\lambda kT} - 1} \quad ; \lambda_m = 2000\,\text{Å}$$

Maximum power at λ_m satisfies $\frac{hc}{\lambda_m kT} = 4.96$

(a) $\lambda = 4000\,\text{Å} = 2\lambda_m$. Power at $\lambda_m = 1\,\mu W$

$$\frac{u(\lambda)}{u(\lambda_m)} = \frac{\lambda_m^5}{\lambda^5} \frac{e^{hc/\lambda_m kT} - 1}{e^{hc/2\lambda_m kT} - 1} = \frac{1}{2^5} \frac{e^{4.96} - 1}{e^{2.48} - 1} = 0.40$$

$\Rightarrow$ power emitted at $4000\,\text{Å} = \boxed{0.40\,\mu W}$

(b) $\lambda = 1000\,\text{Å} = \lambda_m/2$

$$\frac{u(\lambda)}{u(\lambda_m)} = 2^5 \frac{e^{4.96} - 1}{e^{9.92} - 1} = 0.22$$

$\Rightarrow$ power emitted at $1000\,\text{Å} = \boxed{0.22\,\mu W}$

(c) For large λ, $u(\lambda) = \frac{8\pi}{\lambda^4} k_B T \Rightarrow$

$$u(\lambda) = u(\lambda_m) \cdot \left(\frac{\lambda_m}{\lambda}\right)^4 \cdot k_B T \frac{e^{4.96} - 1}{\frac{hc}{\lambda_m}} = u(\lambda_m)\left(\frac{\lambda_m}{\lambda}\right)^4 \frac{e^{4.96} - 1}{4.96}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda_m}\right)^4 = \frac{u(\lambda_m)}{u(\lambda)} \frac{e^{4.96} - 1}{4.96} \Rightarrow \lambda = \lambda_m \cdot \left(10^4 \frac{e^{4.96} - 1}{4.96}\right)^{1/4}$$

$$\Rightarrow \boxed{\lambda = 23.1\,\lambda_m = 46{,}230\,\text{Å}}$$

Problem 3

$H_{op}\Psi = E\Psi$ is satisfied for all x

We have $E = \frac{\hbar\omega}{2} \Rightarrow$ on left side, Ψ = ground state of harmonic oscillator wave function $\Rightarrow$

$\hbar\omega = 2eV$

$\Psi(x) = e^{-\lambda x^2}$ for $x<0$.

For $x>0$: $\Psi(x) = A\cos(kx) + B\sin(kx)$

energy is $\frac{\hbar^2 k^2}{2m_e} = \frac{\hbar\omega}{2} \Rightarrow k = \left(\frac{\hbar\omega}{\hbar^2/m_e}\right)^{1/2} = \left(\frac{2eV}{7.62\,eV\,\mathring{A}^2}\right)^{1/2}$

$\Rightarrow \boxed{k = 0.51\,\mathring{A}^{-1}}$

Continuity of Ψ and Ψ'

$\Psi(x=0) = 1 = A$

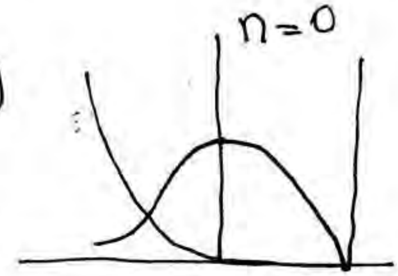
$\Psi'(x=0) = -2\lambda x e^{-\lambda x^2} = 0 = -kA\sin(k\cdot 0) + kB\cos(0)$

$\Rightarrow B = 0 \Rightarrow$

$\Psi(x) = \cos(kx)$ on right.

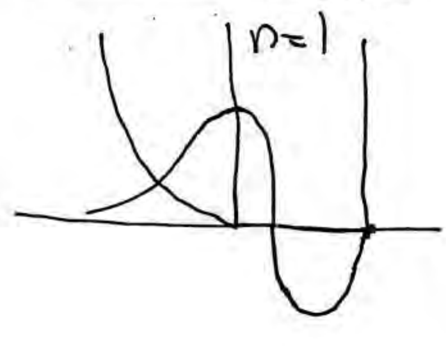
$\Psi(x=L) = 0 \Rightarrow k = \frac{\pi}{L}\left(n+\frac{1}{2}\right) \Rightarrow$

$\Rightarrow L = \frac{\pi(n+\frac{1}{2})}{k} = 6.13\,\mathring{A}\left(n+\frac{1}{2}\right)$



(a) Smallest L : n=0

$\Rightarrow \boxed{L = 3.07\,\mathring{A}}$



(b) next smallest L: n=1

$\Rightarrow \boxed{L = 9.20\,\mathring{A}}$

Problem 4

$$E = \frac{\hbar^2 \pi^2}{2 m_e L^2} (n_1^2 + n_2^2) \equiv E_0 (n_1^2 + n_2^2)$$

		$n_1^2+n_2^2$
(2,3),(3,2)	— —	13
(1,3),(3,1)	↑↓ —	10
(2,2)	↑↓	8
(1,2)(2,1)	↑↓ ↑↓	5
(1,1)	↑↓	2

$$E_0 = \frac{\hbar^2 \pi^2}{2 m_e L^2} = 1.045 \text{ eV}$$

Ground state energy for 10 electrons:

$$E_{GS} = E_0 \cdot 2 \cdot (2 + 2 \cdot 5 + 8 + 10) = 60 E_0$$

$$\Rightarrow \boxed{E_{G.S.} = 62.7 \text{ eV}}$$

(b) Lowest energy transition → longest wavelength photon

$= (2,2)$ to $(3,1)$ $\Rightarrow n_1^2 + n_2^2 = 8$ to $n_1^2 + n_2^2 = 10 \Rightarrow$

$$\Delta E = 2 E_0 = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{2 E_0}$$

$$\Rightarrow \boxed{\lambda = 5933 \text{ Å}}$$

Problem 5

$$L^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$F(\theta,\phi) = (\cos^2\theta + a)\, e^{i\alpha\phi}$$

$$\frac{\partial F(\theta,\phi)}{\partial\theta} = -2\cos\theta\sin\theta\, e^{i\alpha\phi} \quad ; \quad \frac{\partial^2 F}{\partial\phi^2} = -\alpha^2(\cos^2\theta + a)e^{i\alpha\phi}$$

$$\frac{\partial^2 F}{\partial\theta^2} = 2(\sin^2\theta - \cos^2\theta)\, e^{i\alpha\phi}$$

$$L^2 F(\theta,\phi) = \lambda F(\theta,\phi) \Rightarrow$$

$$-\hbar^2\left[2(\sin^2\theta - \cos^2\theta) - 2\cos^2\theta - \frac{\alpha^2(\cos^2\theta + a)}{\sin^2\theta}\right] = \lambda(\cos^2\theta + a)$$

Using $\sin^2\theta = 1 - \cos^2\theta$

$$-\hbar^2 \cdot 2 \cdot \left[1 - 3\cos^2\theta - \frac{\alpha^2(\cos^2\theta + a)}{2\sin^2\theta}\right] = \lambda(\cos^2\theta + a)$$

This can be satisfied in 2 ways:

(a) $\alpha = 0 \Rightarrow 6\hbar^2(\cos^2\theta - \frac{1}{3}) = \lambda(\cos^2\theta + a) \Rightarrow$

$\Rightarrow \boxed{a = -\frac{1}{3}, \quad \lambda = 6\hbar^2}$

(b) $a = -1 \Rightarrow \cos^2\theta + a = \cos^2\theta - 1 = -\sin^2\theta \Rightarrow$

$-\hbar^2 \cdot 2\left[1 - 3\cos^2\theta + \frac{\alpha^2}{2}\right] = \lambda(\cos^2\theta + a) \Rightarrow 6\hbar^2\left[\cos^2\theta - \frac{1}{3} - \frac{\alpha^2}{6}\right] = \lambda(\cos^2\theta + a)$

$\Rightarrow \boxed{\lambda = 6\hbar^2}, \quad \frac{1}{3} + \frac{\alpha^2}{6} = -a = 1 \Rightarrow \frac{\alpha^2}{6} = \frac{2}{3} \Rightarrow \boxed{\alpha = \pm 2}$

(a) is $l = 2, m_l = 0$; (b) is $l = 2, m_l = \pm 2$

Problem 6

Energies of 3d harmonic oscillator

$$E_{n_1 n_2 n_3} = \frac{3\hbar\omega}{2} + \hbar\omega\,(n_1 + n_2 + n_3)$$

Ground state energy: $E_{000} = \frac{3\hbar\omega}{2} = 0.06\ \text{eV} \Rightarrow$

$$\Rightarrow \boxed{\hbar\omega = 0.04\,\text{eV}}$$

Selection rules: n_i can only change by 1 $\Rightarrow$ there is only one photon wavelength: $\frac{hc}{\lambda} = \hbar\omega$

$$\boxed{\lambda = \frac{hc}{\hbar\omega} = 310{,}000\ \text{Å}}$$

(b) In the presence of a magnetic field, states with $l \neq 0$ shift energy. The first excited states can be written as eigenstates of L^2, L_z, with $l=1$, $m_l = \pm 1, 0$ (see hw problem 8).

$$\Delta E = \mu_B B m_l = 5.21 \times 10^{-3}\, m_l\ \text{eV}$$

Selection rules are $\Delta l = \pm 1$, $\Delta m_l = \pm 1, 0$. In the ground state $l = 0$, $m_l = 0$ $\Rightarrow$ we get 3 lines:

to state $l=1$, $m_l = 0$: $\Delta E = 0.04\,\text{eV}$, $\lambda = 310{,}000\ \text{Å}$

" " $l=1$, $m_l = 1$: $\Delta E = 0.0452$, $\lambda = 274{,}276\ \text{Å}$

" " $l=1$, $m_l = -1$: $\Delta E = 0.0348$, $\lambda = 356{,}424\ \text{Å}$

Problem 7 $\quad g(E) = CVE^{1/2}$

$$N = \int_0^{E_F} dE\, g(E) = CV \cdot \frac{2}{3} E_F^{3/2} \Rightarrow$$

$$\Rightarrow \boxed{E_F = \left(\frac{3}{2C} \frac{N}{V}\right)^{2/3}}$$

(b) $$E_T = \int_0^{E_F} dE\, E\, g(E) = CV \cdot \frac{2}{5} E_F^{5/2}$$

replacing $CV = \frac{3}{2} \frac{N}{E_F^{3/2}} \Rightarrow \quad E_T = \frac{3}{2} \cdot \frac{2}{5} N \frac{E_F^{5/2}}{E_F^{3/2}}$

$$\Rightarrow \boxed{E_T = \frac{3}{5} N E_F}$$

(c) $$P = -\frac{dE_T}{dV} = -\frac{3}{5} N \frac{dE_F}{dV}$$

$$\frac{dE_F}{dV} = -\frac{2}{3} \frac{E_F}{V} \Rightarrow P = \frac{2}{3} \cdot \frac{3}{5} \frac{N E_F}{V} = \frac{2}{5} \frac{N k_B T_F}{V}$$

$$\Rightarrow \boxed{PV = \frac{2}{5} (nRT_F)}$$

Problem 8

Charecteristic energy of rotation is $E_R = \frac{\hbar^2}{2I}$

$I = \frac{1}{2} M R^2$. $R = 1.08 \mathring{A}$. $M \sim 4 \times 939 \frac{MeV}{c^2} \Rightarrow$

$$E_R = \frac{\hbar^2}{MR^2} = \frac{\hbar^2}{939 \frac{MeV}{c^2}} \frac{1}{4R^2} = \frac{\hbar^2}{m_e} \frac{m_e}{939 \frac{MeV}{c^2}} \frac{1}{4R^2} =$$

$$= 7.62 \times \frac{0.511}{939} \times \frac{1}{4 \times 1.08^2} eV \Rightarrow E_R = 0.000889 eV$$

$$\Rightarrow \boxed{T_R = \frac{E_R}{k_B} = 10.3 K}$$

(b) The peaks in the absorption spectrum result from transitions from $V=0$ to $V=1$ (vibrational) and l to $l+1$ or $l-1$ (rotational).

For example, $\Delta E_{l \to l+1} = \hbar \omega + 2(l+1) E_R$

next peak: $\Delta E_{l+1 \to l+2} = \hbar \omega + 2(l+2) E_R$

distance between peaks $\Delta E_{l+1 \to l+2} - \Delta E_{l \to l+1} = 2 E_R = \boxed{0.0018 eV}$

(c) Thermal occupation $n(E_l) = (2l+1) e^{-l(l+1) E_R / k_B T}$

Maximum: $\frac{dn}{dl} = 0 = 2 - (2l+1)^2 \frac{E_R}{k_B T} \Rightarrow$

$$\Rightarrow 2l+1 = \left(\frac{2k_B T}{E_R}\right)^{1/2} \Rightarrow l = \frac{1}{2}\left(\left(\frac{2k_B T}{E_R}\right)^{1/2} - 1\right) = \frac{1}{2}\left(\left(\frac{2T}{T_R}\right)^{1/2} - 1\right) = 3.3$$

$$\Rightarrow \boxed{l = 3}$$