Stochastic population genetics: homework 6
To be returned on June 7

June 2, 2017

1 Two-loci dynamics

In this problem we are interested in the dynamics of a large population of diploids (throughout the exercise we will always consider the limit of infinite population). We consider two loci in the genome: the first one has two possible alleles A and a, and the second one has two possible alleles B and b. The probabilities of the four types AB, Ab, aB, ab, are x_1, x_2, x_3 and x_4,respectively. We also define p_1, q_1, p_2 and q_2 respectively the probability of alleles A, a, B and b. Finally, we define linkage disequilibrium as

$$D = x_1 - p_1 q_1. \tag{1}$$

a. Express $\{p_i\}$ and $\{q_i\}$ in terms of the $\{x_i\}$. Check that you can rewrite D as $x_1 x_4 - x_2 x_3$. If alleles are combined into gametes randomly and the system evolves over a long time, what should be the value of D?

b. We define the indicator random variable l_1 as 1 if the allele is A and 0 if the allele is a. We define equivalently l_2 for alleles B and b. Show that

$$D = \mathrm{cov}(l_1, l_2). \tag{2}$$

Check that the p and qs are conserved.

c. We now introduce recombination: when alleles WX/YZ produce gametes, they will produce gametes WX and YZ with probability $1-r$ (no recombination) and gametes WZ and YX with probability r (recombination). Show that

$$D_t = D_0 (1-r)^t. \tag{3}$$

What is the typical time over which the system's linkage disequilibrium goes to 0?

We now introduce selection in the population: allele i/j has fitness $w_{i,j}$ where $(i,j) \in [\![1,4]\!]^2$ and i or j equals $1,2,3,4$ correspond respectively to AB, Ab, aB and ab. Assuming symmetrical maternal and paternal influence on fitness and that there is no cis-trans effect, i.e. $w_{ij} = w_{ji}$ and $w_{23} = w_{14}$, we can rewrite the fitness as a function of only nine coefficients.

d. Show that

$$x_1^{(t+1)} = \frac{x_1^{(t)}(\sum_i w_{1i} x_i^{(t)}) - r w_{14} D_t}{\bar{w}^{(t)}}, \tag{4}$$

where $\bar{w}^{(t)}$ is the average fitness at time t. What is the equation for the other x_i?

e. Start from the equilibrium point where all the population if AB/AB. By introducing a small fraction of the population ϵ with a different genotype, determine the condition for the stability of the monomorphic type AB/AB.