Stochastic population genetics: homework 2
To be returned on May 8

April 28, 2017

1 Demographic noise

In this exercise we consider a population of clones (individuals with the same genetic code) evolving in time. $N(t)$ is the number of individuals in the population at time t. At each time step δt, the population can increase by one unit through division with probability $N\nu\delta t$ or it can decrease by one unit through death with probability $N\mu\delta t$. Every time the the population goes extinct, i.e. the process hits zero, the process is restarted from $N = N_0$. The death rate is greater than the reproduction rate, i.e. $\mu > \nu$.

a. Write down the master equation for $P(N, t + \delta t)$ in terms of $P(N, t)$, $P(N + 1, t)$ and $P(N - 1, t)$. Write down explicitly the relation for the special indices $N = 1$ and $N = N_0$, where a source term is present.

b. The index $N = N_0$ has a source term, which reflects the restarting mentioned above. Express the amplitude of the source. Verify that the total probability $\sum_{i=1}^{\infty} P(N, t)$ is conserved in time.

c. At the steady state, for large N one expects an exponential decays $P(N) \propto \lambda^N$. Use the master equation derived in (a) to calculate the rate of decay λ.

d. Using results derived in class (and assuming that moments higher than 2 are negligible), show that the Fokker-Planck equation for the process in the continuous time limit can be written as

$$\partial_t \rho(N, t) = (\mu - \nu)\partial_N (N\rho(N, t)) + \frac{1}{2}(\mu + \nu)\partial_N^2 (N\rho(N, t)) + \delta(N - N_0)s_N, \quad (1)$$

where δ is the Dirac distribution and s_N is the rate of the process' restarting.

e. What is the relation between s_N and the flux of probability J_0 at the boundary 0?

f. Solve the steady-state Fokker-Planck equation for large N. In what limit are the decay exponents of the discrete and continuous solutions equivalent ?

g. We now consider the process without a source and want to compute the statistics of the first exit time through the absorbing barrier at 0. Use what you learnt in class to write down the equation for the current of probability J_0 at 0. What is the mean first passage time of the process in 0? Discuss the limit of $\mu - \nu = 0$.