II.) Technical Aside: IMPORTANT

- Where does CGL come from?
- Why is it so *!?√X ubiquitous?

Consider nonlinear oscillator:

$$\underbrace{\ddot{x} + \omega_0^2 x}_{\text{oscillator}} = \underbrace{f(x, \dot{x})}_{\text{nonlinearity}} + \underbrace{\epsilon\, p(t)}_{\text{forcing} \to \text{frequency } \omega}$$

seek: $x(t) = \frac{1}{2}\left(\underbrace{A(t)}_{\text{amplitude}} \underbrace{e^{i\omega t}}_{\text{"entraining" frequency}} + \text{c.c.}\right)$

(not necessarily slow ⟹ i.e. phase jumps

Now, then convenient to re-write:

$$\ddot{x} + \omega^2 x = (\omega^2 - \omega_0^2)x + f(x_0, \dot{x}) + \epsilon p(t)$$

or $\quad \dot{x} = y$

$$\dot{y} = -\omega^2 x + (\omega^2 - \omega_0^2)x + f(x, y) + \epsilon p(t)$$

so $\underline{\underline{\text{if}}}$ $\quad y = \frac{1}{2}\left(i\omega A(t)\, e^{i\omega t} + c.c.\right)$

$(y = \dot{x})$

then: $\quad$ Amplitude Eqn $\Leftrightarrow$ Complex A

$$* \quad \boxed{\underbrace{\dot{A}(t)}_{\text{Amplitude}} = \frac{e^{-i\omega t}}{i\omega}\left[\underbrace{(\omega^2 - \omega_0^2)x}_{\text{mis-match}} + \underbrace{f(x, y)}_{\text{nonlinearity}} + \underbrace{\epsilon p(t)}_{\text{forcing}}\right]}$$

Now, as usual:

- interested in $\underline{\text{slowest}}$, largest variations on RHS
  $\Rightarrow$ isolate slowest terms

- eliminate fast oscillations via averaging
  (akin method of averaging)

To Average:

- subsitute x, y in terms $A(t)$ on RHS of $*$
- neglect oscillating terms (on ω scale)

$$\dot{A} = \frac{e^{-i\omega t}}{i\omega}\Big[\underset{①}{(\omega^2-\omega_0^2)x} + \underset{②}{f(x,y)} + \underset{③}{\epsilon p(t)}\Big]$$

①:

$$\dot{A}_{①}(t) = \int_0^{2\pi/\omega} d\tau \frac{e^{-i\omega\tau}}{i\omega}\left[(\omega^2-\omega_0^2)\left(\tfrac{1}{2}A(t)e^{i\omega\tau} + \text{c.c.}\right)\right]$$

$$= \frac{(\omega^2-\omega_0^2)}{2i\omega}A(t)$$

②:

$$A_{②}(t) \equiv \int_0^{2\pi/\omega} d\tau \frac{e^{-i\omega\tau}}{i\omega} f(x,y)$$

Now, $f(x,y)$ is nonlinear, in general,

so

$$f(x,y) = \sum_{m,n} C_{m,n} \left(Ae^{i\omega t}\right)^n \left(A^* e^{-i\omega t}\right)^m$$

$\Rightarrow$

$$A_{(2)} = \int_0^{2\pi} d\tau \frac{e^{-i\omega\tau}}{i\omega} \sum_{m,n} C_{m,n} \left(Ae^{i\omega t}\right)^n \left(A^* e^{-i\omega t}\right)^m$$

only $n - m - 1 = 0$ don't vanish (extracts phase coherent piece of nonlinearity coherent with $e^{i\omega t}$)

$$m = n-1$$

Thus $A_{(2)}$ must have form:

$$A_{(2)} = \underbrace{g(|A|^2)}_{\text{arbitrary (set by problem)}} A \qquad \left(\text{i.e. only } \updownarrow \text{ phase}\right)$$

simplest choice: $g(|A|^2) = \mu - \gamma' |A|^2$

$$g(|A|^2)A = \underbrace{\mu A}_{\substack{\text{linear}\\ \text{term}\\ \text{(growth)}}} - \underbrace{(\gamma + i\kappa)|A|^2 A}_{\substack{\text{lowest n.l. term } \Rightarrow\\ \text{(saturation)}}}$$

$\mu, \gamma > 0 \Rightarrow$ supercritical bifurcation

$\mu, \gamma < 0 \Rightarrow$ sub-critical bifurcation $\Rightarrow$ need h.o. to saturate.

Similarly, as have:

$$\rho(t) = \sum_n \left(\rho_n e^{in\omega t} + c.c.\right)$$

$$\int_0^{2\pi/\omega} d\tau \frac{e^{-i\omega t}}{i\omega} \epsilon \rho(t) = -i\epsilon E$$

so

$$\dot{A} = \underbrace{-i\frac{(\omega^2-\omega_0^2)}{2\omega}A}_{\text{mismatch}} + \underbrace{\mu A}_{\text{growth}} \underbrace{-(\gamma+iR)|A|^2A}_{\substack{\text{NL saturation} \\ \text{NL freq shift}}} \underbrace{-i\epsilon E}_{\text{drive}}$$

$\Rightarrow$ recovers CGL structure!

Note:

- derivation is "generic" to form of nonlinear oscillator.
- in general: $g(|A|^2) = \sum_n g_n (|A|^2)^n$

  with coeffs set by problem.

- not surprisingly, can also describe via method of reductive perturbation theory (i.e. Poincaré-Lindstedt).

- in absence of forcing, recovers Landau-Stuart:

$$\frac{dA}{dt} = (1+i\eta) A - (1+i\alpha)|A|^2 A$$

- for validity, need:

$$\left. \begin{array}{l} |\omega-\omega_0| \ll \omega_0 \\ \mu \ll \omega_0 . \end{array} \right\} \text{ ensure } \begin{array}{l} \text{— NL term \underline{small}} \\ \text{— \underline{weak} instability of } A=0 \text{ fixed point} \end{array}$$

Note: Story is consistent, i.e.

μA vs $\gamma |A|^2 A$ ensures:

<u>so</u> $|A|^2 \lesssim \mu/\gamma \Leftrightarrow NL \sim L$

$\Leftrightarrow$ requires small growth. In practice, CGL valid only <u>near</u> marginality.