$\Rightarrow$

Two Interacting Oscillators - Weak Interaction Case.

Now, - recall single oscillator plus forcing

$$d\phi/dt = \omega_0 + \epsilon Q(\phi, t) \quad \rightarrow \begin{matrix}\omega \\ \omega_0\end{matrix}$$

- consider <u>two</u> interacting oscillators

<u>i.e.</u>

- replace external forcing by 2^{nd} oscillator
- continue in vein of phase dynamics.

Now, have 2 self-sustained oscillators of form:

$$\frac{d\phi_1}{dt} = \omega_1 \ , \quad \frac{d\phi_2}{dt} = \omega_2 \qquad \left.\begin{matrix}\omega_1 \\ \omega_2\end{matrix}\right\} \text{limit cycle frequencies}$$

<u>So</u>, no loss of generality to immediately write for phase dynamics:

$$\left.\begin{aligned}\frac{d\phi_1}{dt} &= \omega_1 + \epsilon Q_1(\phi_1, \phi_2) \\ \frac{d\phi_2}{dt} &= \omega_2 + \epsilon Q_2(\phi_1, \phi_2)\end{aligned}\right] \text{coupling functions.}$$

- $Q_{1,2}$ are 2π periodic in ϕ_1, ϕ_2
- cycles ϕ_1, ϕ_2 define 2D toroidal invariant surface, for motion.

For phase dynamics can proceed a/a single oscillator plus force:

$$Q_1(\phi_1, \phi_2) = \sum_{k,l} a_1^{k,l} e^{ik\phi_1} e^{il\phi_2}$$

$$Q_2(\phi_2, \phi_1) = \sum_{k,l} a_2^{l,k} e^{ik\phi_1 + il\phi_2}$$

$$\phi_1 = \omega_1 t$$
$$\phi_2 = \omega_2 t$$

$$\exp[ik\phi_1 + il\phi_2] = \exp[i(\phi_0 + i(k\omega_1 + l\omega_2)t]$$

$$k\omega_1 + l\omega_2 \approx 0 \quad \Rightarrow \begin{cases} \text{resonant contribution} \\ \text{slow, DC forcing} \end{cases}$$

Further take: $\frac{\omega_1}{\omega_2} \approx \frac{m}{n}$

so resonance for $k = nj$
$\ell = -mj$

so can write:

$$\begin{cases} \dfrac{d\phi_1}{dt} = \omega_1 + \epsilon q_1(n\phi_1 - m\phi_2) \\ \dfrac{d\phi_2}{dt} = \omega_2 + \epsilon q_2(m\phi_2 - n\phi_1) \end{cases}$$

and
$$\begin{cases} q_1(n\phi_1 - m\phi_2) = \sum_j a_1^{nj,-mj} e^{ij(n\phi_1 - m\phi_2)} \\ q_2(m\phi_2 - n\phi_1) = \sum_j a_2^{mj,-nj} e^{ij(m\phi_2 - n\phi_1)} \end{cases}$$

Now, define difference between phases:

$$\boxed{\begin{aligned} &\psi \equiv n\phi_1 - m\phi_2 \\ &\frac{d\psi}{dt} = -\nu + \epsilon q(\psi) \end{aligned}} \Rightarrow \begin{cases} \nu = m\omega_2 - n\omega_1 \\ q(\psi) \equiv n q_1(\psi) - m q_2(-\psi) \end{cases}$$

$\downarrow$

reduces to 1 oscillator synchronization problem.

Next: Beyond Phase Dynamics!

Dynamical System (Dissipative/Nonlinear)
$\Rightarrow$ Limit Cycle

$\downarrow$ Reductive P.T.
Method of Averaging

CGL Equation (or coupled)

$\downarrow$ $Re^{i\theta}$

Amplitude
Phase } Equations

$\downarrow$ $R = 1 + \hat{r}$
$\dot{\hat{r}} = -2\hat{r} + \cdots$ $\Rightarrow$ slave $\hat{r}$ to θ

$\boxed{\text{Phase Equation}}$ $\Rightarrow$ { Focus of Synchronization Theory

→ Amplitude Equations: Coupled Oscillators

Consider 2 weakly nonlinear oscillators:

$$\ddot{x}_1 + \omega_1^2 x_1 = f_1(x_1, \dot{x}_1) + D_1(x_2 - x_1) + B_1(\dot{x}_2 - \dot{x}_1)$$

$$\ddot{x}_2 + \omega_2^2 x_2 = f_2(x_2, \dot{x}_2) + D_2(x_1 - x_2) + B_2(\dot{x}_1 - \dot{x}_2)$$

- linear coupling
- difference coupling ↔ "diffusive"
  (anticipates phase diffusion)

but also can be... ↔ "direct" coupling

i.e. $RHS_1 = D_1 x_2 + B_1 \dot{x}_2$

Aim: Link between structure of coupling and macro-phenomena (i.e. oscillation death)

As before, $(x, y)_{1,2} = \begin{pmatrix} 1/2\, A_{1,2}(t)\, e^{i\omega t} + c.c. \\ i\omega\,(\quad) \end{pmatrix}$

⇒ amplitude equations via averaging ⇒

$$\begin{cases} \dot{A}_1 = -i\Delta_1 A_1 + \mu_1 A_1 - (\gamma_1 + i\alpha_1)|A_1|^2 A_1 + (\beta + i\delta)(A_2 - A_1) \\ \dot{A}_2 = -i\Delta_2 A_2 + \mu_2 A_2 - (\gamma_2 + i\alpha_2)|A_2|^2 A_2 + (\beta_2 + i\delta)(A_1 - A_2) \end{cases}$$

$\delta_{1,2} \to$ reactive (ω effect)

i.e. $\text{Coupling}_1 = (\beta_1 + i\delta_1)(A_2 - A_1)$

$\hookrightarrow \beta_{1,2} \to$ dissipative

$\text{Coupling}_2 = (\beta_2 + i\delta_2)(A_1 - A_2)$

$\Delta_{1,2} = \omega_{1,2} - \omega \to$ mis-match.

Now, to save algebra:

- $A_{1,2} = R_{1,2}\, e^{i\phi_{1,2}}$ (Amplitude Phase Rep.)

- $\psi = \phi_2 - \phi_1$ (via difference coupling)

$\Rightarrow$

$$\partial R_1/\partial t = \mu_1 R_1 (1 - \gamma_1 R_1^2) + \beta_1 (R_2 \cos\psi - R_1) - \delta_1 R_2 \sin\psi$$

$$\partial R_2/\partial t = \mu_2 R_2 (1 - \gamma_2 R_2^2) + \beta_2 (R_1 \cos\psi - R_2) + \delta_2 R_1 \sin\psi$$

$$\partial \psi/\partial t = -\nu + \mu\alpha_1 R_1^2 - \mu_2 \alpha_2 R_2^2$$

$$+ \left(\delta_2 \frac{R_1}{R_2} - \delta_1 \frac{R_2}{R_1}\right)\cos\psi + \delta_1 - \delta_2$$

$$- \left(\beta_1 \frac{R_2}{R_1} + \beta_2 \frac{R_1}{R_2}\right)\sin\psi$$

Further: clears system
$$\begin{cases} \mu_1 = \mu_2 = \mu \\ t \to t/\mu \\ A \to A/(\gamma/\mu)^{1/2} \\ \beta, \delta \to \text{normalized to } \mu \\ \alpha \to \text{normalized to } \gamma/\mu \end{cases}$$

$\Rightarrow$

$$\begin{cases} \dot{R}_1 = R_1(1-R_1^2) + \overset{*}{\beta}(R_2\cos\psi - R_1) - \overset{*}{\delta} R_2 \sin\psi \\ \dot{R}_2 = R_2(1-R_2^2) + \overset{*}{\beta}(R_1\cos\psi - R_2) + \overset{*}{\delta} R_1 \sin\psi \\ \dot{\psi} = -\nu + \overset{*}{\alpha}(R_1^2 - R_2^2) + \overset{*}{\delta}\left(-\frac{R_2}{R_1} + \frac{R_1}{R_2}\right)\cos\psi \\ \qquad - \overset{*}{\beta}\left(\frac{R_2}{R_1} + \frac{R_1}{R_2}\right)\sin\psi \end{cases}$$

Phase and 2 Amplitude System:

$\alpha \to$ NL frequency shift
$\not\to$ $\alpha = 0$ "isochronous" (new use)

$\nu \to$ frequency detuning

$\delta \to$ reactive coupling

$\beta \to$ dissipative coupling

Now, consider phenomena exhibited by the system

a.) oscillation death / quenching

b.) attractive / repulsive interaction

a.) Oscillation Death

→ large β, γ ⇒ $R_1 = R_2 = 0$ becomes stable.

∴

→ oscillations die.

To see:
- $\gamma \equiv 0$, dissipative coupling only (via β)
- $\omega \equiv (\omega_1 + \omega_2)/2$ so
  $\Delta_1 = -\Delta_2 = \Delta$

and obtain, for amplitude equation:

$$\dot{A}_1 = (i\Delta + \mu) A_1 + \beta (A_2 - A_1) + \cancel{NL}$$

$$\dot{A}_2 = (-i\Delta + \mu) A_2 + \beta (A_1 - A_2) + \cancel{NL}$$

i.e. perturb about $A_1 = A_2 = 0$

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_{1,0} \\ A_{2,0} \end{pmatrix} e^{\lambda t}$$

$$\Rightarrow \quad \boxed{\lambda = \mu - \beta \pm \sqrt{\beta^2 - \Delta^2}}$$

Need : $\left[\begin{array}{l} \lambda < 0 \\ \mu < \beta \quad \text{and} \quad \beta < (\mu^2 + \Delta^2)/2\mu . \end{array}\right.$

Key: ① $\underbrace{\mu < \beta}$

② $\underbrace{\beta < \frac{\Delta^2}{2\mu}} + \dots$

① → "diffusive" coupling brings additional "difference" dissipation to each oscillator.
(i.e. each 'drags other down)'

② → detuning is large enough so forcing from other oscillator can't excite

b.) Attractive / Repulsive Interaction

- reduce to phase description
- derive directly; for β, δ small

excursion

so $\quad R_{1,2} \approx 1 + r_{1,2}$ (perturb about oscillator)

$r_{1,2} \ll 1$

$\Rightarrow$ plugging in to $\dot{R}_1$, $\dot{R}_2$ and linearizing $\Rightarrow$

$$\dot{r}_1 = -2r_1 + \beta(\cos\psi - 1) - \delta\sin\psi$$

$$\dot{r}_2 = -2r_2 + \beta(\cos\psi - 1) + \delta\sin\psi$$

strong damping $\Rightarrow$ $\dot{r}_1 = \dot{r}_2 = 0$

$$\therefore \quad r_{\substack{1\\2}} = \frac{\beta}{2}(\cos\psi - 1) \mp \frac{\delta}{2}\sin\psi$$

$$R_{1,2} = 1 + r_{1,2}$$

and plugging into phase equation:

$$\psi = \phi_2 - \phi_1$$

$$\boxed{\dot{\psi} = -\nu - 2(\beta + \alpha\delta)\sin\psi}$$

phase dynamics equation!

Attractive + Repulsive Interaction

Aside: if $q(\psi) = \sin\psi$

$$\frac{d\psi}{dt} = -\gamma + \epsilon \sin\psi$$

so ① $\epsilon < 0 \Rightarrow$ stable f.p. (ψ_{synch}) on $-\pi/2 < \psi < \pi/2$

i.e. $$\frac{d\delta\psi}{dt} = \epsilon \cos\psi_s \, \delta\psi$$

so $\gamma \to 0$ $\quad \psi_s = 0$

stable phase difference Zero

$\Rightarrow$ phases "attract"

② $\epsilon > 0 \Rightarrow$ stable f.p. (ψ_{synch}) on $\pi/2 < \psi < 3\pi/2$

so $\gamma \to 0$ $\quad \psi_s = \pi$

stable phase difference π

$\Rightarrow$ phases "repel"

Now, clear from before:

if $\nu = 0$

$\beta + \alpha\delta > 0 \Rightarrow \psi = 0$ is stable ψ_s.

$\Rightarrow$ "attraction"

$\beta + \alpha\delta < 0 \Rightarrow \psi = \pi$ is stable ψ_s

$\Rightarrow$ "repulsion"

To interpret:

$\beta \rightarrow B_{1,2} \rightarrow$ dissipative coupling

$\delta \rightarrow D_{1,2} \rightarrow$ reactive $\leftrightarrow$ shift eigenfrequencies

$\therefore$ β - dissipative coupling
- drives 2 oscillators to more homogeneous regime
$\Rightarrow$ 'toward' synchronization via drag on each other
$\Rightarrow$ attraction.

δ - reactive coupling
$\Rightarrow$ no effect on isochronous oscillators ($\nu = 0$)

$\Rightarrow$ non-isochronous oscillators $\Rightarrow$

attractive or repulsive, depending on α's sign.

IV.) Synchronization with Noise

Previously : $\begin{cases} \text{— Background} \\ \text{— R.P.T. Averaging} \end{cases}$

(locking) $\begin{cases} \text{- Phase Dynamics: Oscillator + Forcing} \\ \text{— 2 Coupled Oscillators.} \end{cases}$

⇒ all deterministic

Now : Oscillator with Noise.

i) Basic Phase Dynamics

Now, add noise to phase dynamics equation, i.e.

$$\frac{d\psi}{dt} = \underbrace{-\nu}_{\text{mismatch}} + \underbrace{\epsilon\, q(\psi)}_{\text{forcing}} + \underbrace{\xi(t)}_{\text{noise}}$$

so phase dynamics equation now a Langevin Equation (n.b. multiplicative noise in q possible too)

Convenient to write : $\begin{cases} -\nu + \epsilon q(\psi) = -\dfrac{dV}{d\psi} \\ V(\psi) = \nu\psi - \epsilon\displaystyle\int^{\psi} q(x)\,dx \end{cases}$

defines potential

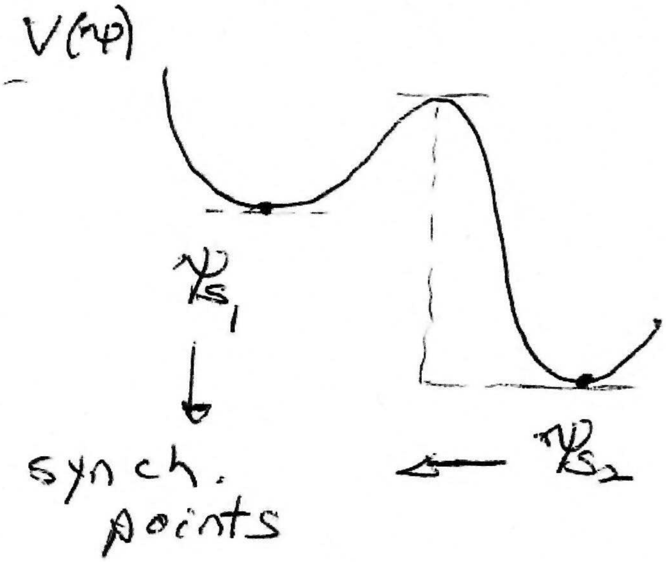
so now have:

$$\boxed{\frac{d\Psi}{dt} = -\frac{dV}{d\Psi} + \varepsilon(t)}$$

$\Bigg\{$ Langevin Equation for particle in potential V

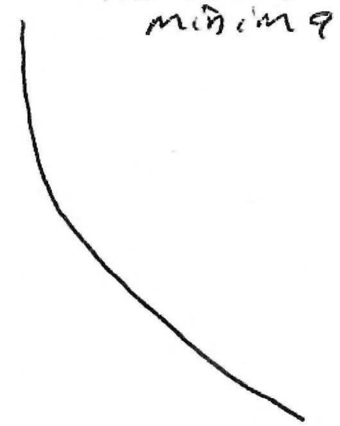
- stable fixed pts $\frac{d^2}{d\psi^2}V > 0$, $\frac{d}{d\psi}V = 0$

- if structured $V(\Psi)$:



vs.

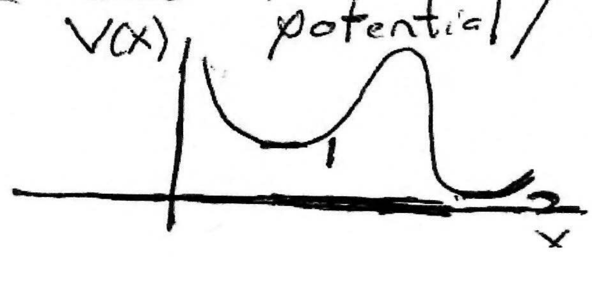
smooth V
no local minima



quasi-periodic motion, only.

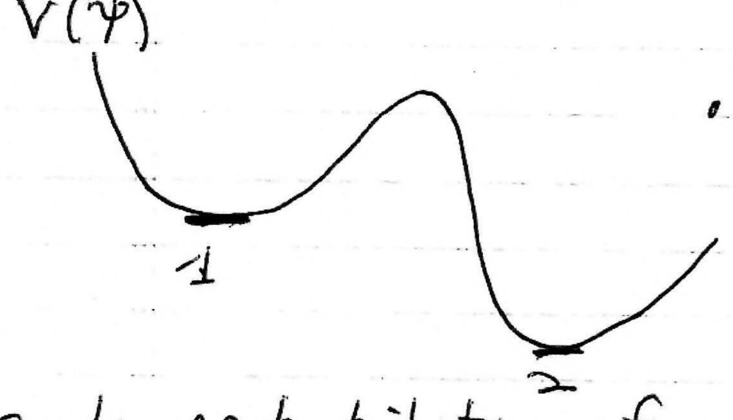
- obvious parallel with Kramers' Problem:

i.e. particle and noise in barrier potential / reaction with activated complex



Flux from 1 → 2 ?
Noise ⟷ kick over barrier

here, have landscape of V in ψ



$\therefore$ transition $\Rightarrow$ noise-induced phase slip.

- seek probability of overcoming barrier $\Rightarrow$ jump in phase of

$$\Delta\psi_{21} = \psi_2 - \psi_1$$

$\leftrightarrow$ analogous J_{12} in Kramers.

- also seek average phase rotation frequency:

$$\Omega_\psi = \langle \dot{\psi} \rangle$$

$$= \int d\psi \, \underbrace{P(\psi)}_{\text{pdf}} \underbrace{\left(-\frac{dV}{d\psi} \right)}_{= \dot{\psi}}$$

, in statistical theory

- important to distinguish between:

- white noise: $S(\omega) = \text{const}$

$$\langle \varepsilon(t_1)\, \varepsilon(t_2) \rangle = \varepsilon_0^2 \, \delta(t_2 - t_1)$$

$\Rightarrow$ Phase kick has Gaussian pdf

$\Rightarrow$ large kicks, slips possible

— $\left\{\begin{matrix}\text{colored} \\ \text{(bounded)}\end{matrix}\right\}$ noise $\rightarrow$ kicks restricted
$\rightarrow$ slips more 'difficult', i.e. only for small barriers

$\rightarrow$? What does synchronization mean in noisey envioronment ?

- need relax $\Omega_\psi = 0$ condition to
- Ω_ψ small but finite...... (see 59a.)
- need statistical treatment to quantify bounds on Ω_ψ.

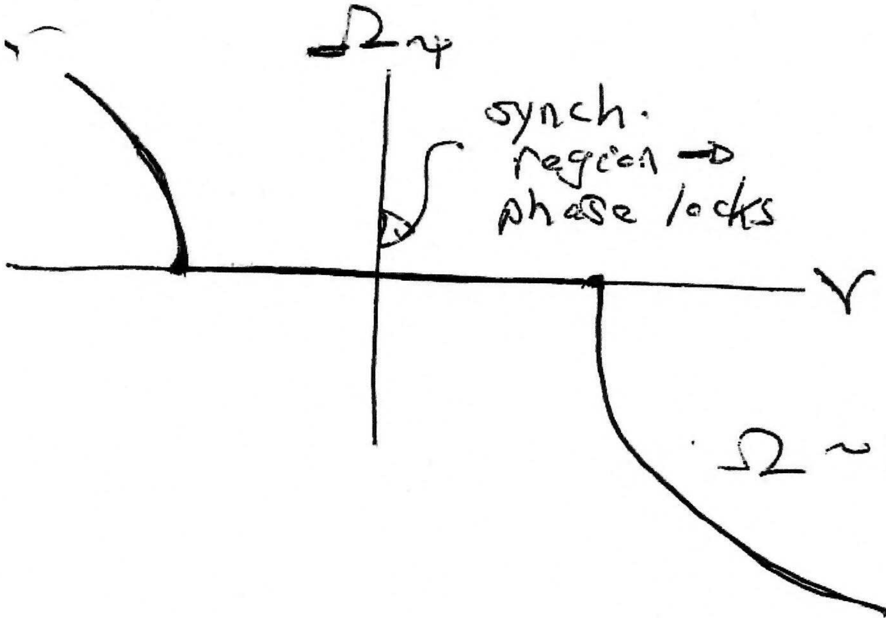
so

ii.) Fokker-Planck Theory $\leftrightarrow$ White Noise

Can immediately write, for $P(\psi)$:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \psi}\left\{ \left\langle \frac{d\psi}{dt} \right\rangle P - \frac{\partial}{\partial \psi} D P \right\}$$

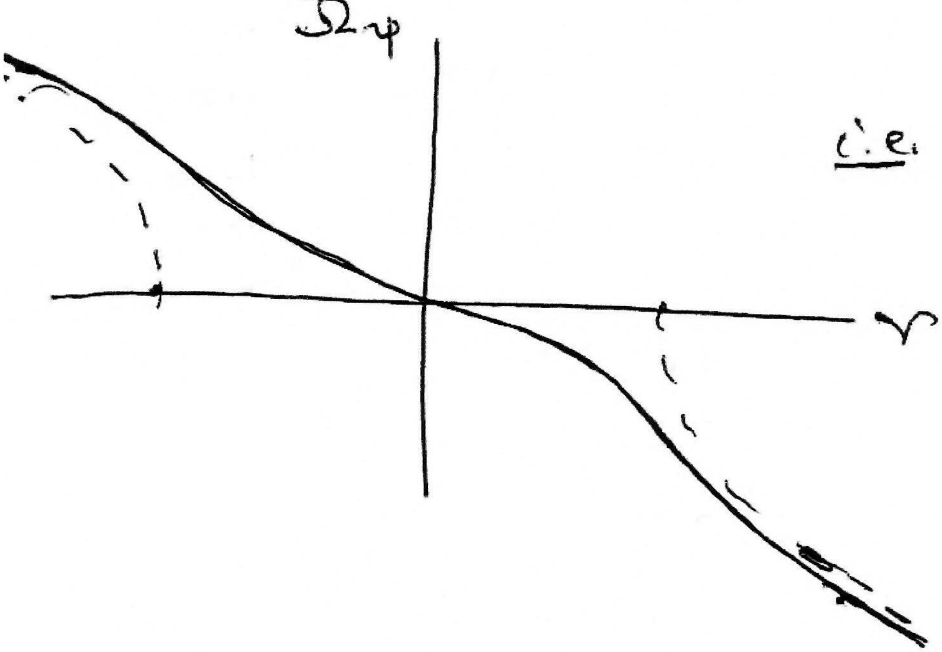
$$D = \frac{\langle \delta\psi \delta\psi \rangle}{2\Delta t} \qquad \delta\psi = \psi - \langle \psi \rangle$$



$\Omega \sim \pm (r - r_M)^{1/2}$

$\varepsilon_0^2 = 0$ case

with noise, expect: (white)



i.e. expect Ω_ψ close to axis, but plateau disappears.

$$\frac{d}{dt}\delta\psi = \varepsilon(t) \qquad \left\langle \frac{d\langle\psi\rangle}{dt} \right\rangle = -\nu + \epsilon g(\psi)$$

$$D = \varepsilon_0^2 = D_0$$

$\hookrightarrow$ cnst.

$\therefore$

$$\boxed{\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \psi}\left\{ (-\nu + \epsilon g(\psi))P - D_0 \frac{\partial P}{\partial \psi} \right\}}$$

or equivalently:

$\Gamma_\psi \equiv$ probability flux

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial \psi}\Gamma_\psi = 0 \quad ; \quad \Gamma_\psi = -\frac{dV}{d\psi}P - D_0\frac{\partial P}{\partial \psi}$$

so have:

$$\langle \Gamma_\psi \rangle = \frac{1}{2\pi}\int_0^{2\pi} \Gamma_\psi \, d\psi, \quad (P \text{ periodic})$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[-\frac{dV}{d\psi}P - \cancel{D_0\frac{\partial P}{\partial \psi}} \right] d\psi$$

$$= \Omega_\psi = \langle \dot{\psi} \rangle$$

$$\boxed{\underbrace{\Omega_\psi}_{\text{slip frequency}} = 2\pi\underbrace{\langle \Gamma_\psi \rangle}_{\text{prob flux}}}$$

To solve F.P.E.:

- stationarity
- periodicity i.e. $P(\psi + 2\pi) = P(\psi)$

$$1/P\,(dP/d\psi) = \frac{-1}{D_0}\frac{dV}{d\psi}\,, \quad \ln P = -\frac{V}{\varepsilon_0^2} + \text{const.}$$

$\therefore$ for periodicity:

$$\begin{cases} P = C\int_{\psi}^{\psi+2\pi} d\psi' \exp\left\{[V(\psi') - V(\psi)]/D_0\right\} \\ C \text{ from } \int P\,d\psi = 1 \end{cases}$$

Now, for Adler equation $\boxed{Z(\psi) = \sin\psi,}$

Convenient to Fourier analyze P, F-P Eqn.:

$$P = \sum_{-\infty}^{+\infty} P_n e^{in\psi}$$

stationarity $\Rightarrow \Gamma_\psi$ independent ψ!

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial \psi}\Gamma_\psi = 0$$

$\therefore$

$$\Gamma_\psi \Big|_{ss} = \Gamma \delta_{n,0}$$

$$\Rightarrow \Gamma \delta_{n,0} = -(in D_0 + r) P_n + \frac{\epsilon}{2i}(P_{n-1} - P_{n+1})$$

i.e. $n=0 \Rightarrow \Gamma = -\gamma P_0 + \frac{\epsilon}{2i}\left(P_{-1} - P_1\right)$

$n \neq 0 \qquad 0 = -(in\,\xi_0^2 + \gamma)P_n + \frac{\epsilon}{2i}\left(P_{n-1} - P_{n+1}\right)$

Further: $\rightarrow$ normalization $\Rightarrow P_0 = 1/2\pi$

$\rightarrow P$ real $\Rightarrow P_{-n} = P_n^*$

$$\Rightarrow \quad \Gamma = -\gamma P_0 + \frac{\epsilon}{2i}\left(P_{-1} - P_1\right)$$
$$= -\gamma/2\pi - \epsilon \,\mathrm{Im}\, P_1$$

$$\Omega_\varphi = 2\pi\langle\Gamma_\varphi\rangle = 2\pi\Gamma$$
$$= -\gamma - 2\pi\epsilon\,\mathrm{Im}\,P_1$$

$\langle\ \rangle \leftrightarrow n=0$ component

$$\boxed{\Omega_\varphi = -\gamma - 2\pi\epsilon\,\mathrm{Im}\,P_1} \rightarrow \text{slip frequency (statistical)}$$

{Flux $\rightarrow$ first harmonic}

To obtain P_1, observe for $n \neq 0$:

$$0 = -(in\,\xi_0^2 + \gamma)P_n + \frac{\epsilon}{2i}\left(P_{n-1} - P_{n+1}\right)$$

$$\boxed{\frac{P_n}{P_{n-1}} = 1 \Big/ \left(r + in\,\mathcal{E}_0^2\right)\frac{2i}{\epsilon} + \frac{P_{n+1}}{P_n}} \Rightarrow$$

continued fraction representation

so

$$\frac{P_1}{P_0} = \frac{1}{\left(r + i\,\mathcal{E}_0^2\right)\frac{2i}{\epsilon} + \frac{P_2}{P_1}}$$

but $\dfrac{P_2}{P_1} = 1 \Big/ \left(r + 2i\,\mathcal{E}_0^2\right)\dfrac{2i}{\epsilon} + \dfrac{P_3}{P_2}$

$\Rightarrow$

$$P_1 = \frac{1/2\pi}{\left(r + i\,\mathcal{E}_0^2\right)\frac{2i}{\epsilon} + \cfrac{1}{\left(r+2i\,\mathcal{E}_0^2\right)\frac{2i}{\epsilon} + \frac{P_3}{P_2}}} \rightarrow$$

continued fraction

$$P_0/P_2 = 1 \Big/ \left(r + 3i\,\mathcal{E}_0^2\right)\frac{2i}{\epsilon} + \frac{P_4}{P_3}$$

etc

so: → flux $\langle \Gamma_\psi^+ \rangle \rightarrow P_1$

→ $P_1 \Leftrightarrow$ continued fraction representation (easily computed for large n)

→ Also interesting to note $P_1 \Leftrightarrow$ Lyapunov exponent of phase dynamics

$$\frac{d\psi}{dt} = -\nu + \epsilon \sin\psi + \tilde{\varepsilon}$$

$$\frac{d}{dt}\delta\psi = \epsilon \cos\psi \, \delta\psi$$

so $$h = \left\langle \frac{1}{\delta\psi}\frac{d\delta\psi}{dt} \right\rangle = \left\langle \frac{d}{dt} \ln \delta\psi \right\rangle$$

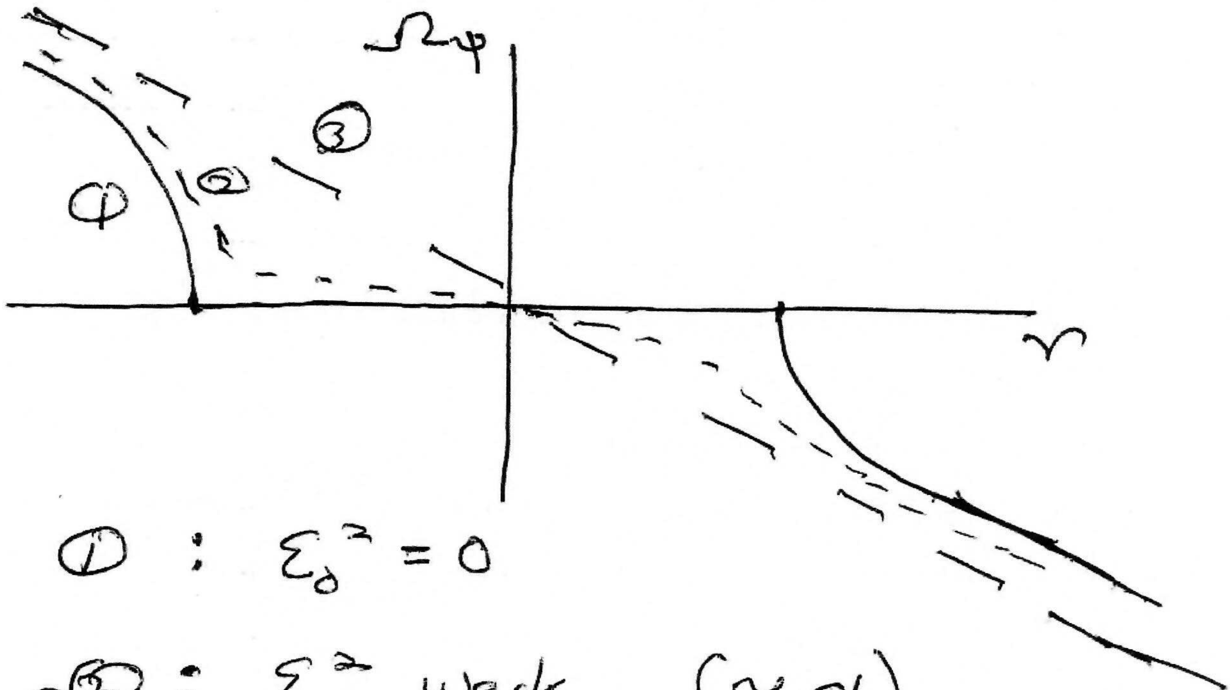
$$= \epsilon \langle \cos\psi \rangle = 2\pi \nu_0 P_1$$

$$\boxed{h = 2\pi \nu_0 P_1}$$

so $\tilde{\varepsilon}_0^2 = 0$, $h = 0$, unless synchronized.

$\tilde{\varepsilon}_0^2 \neq 0$, $h < 0$, all states.

→ Increased noise softens plateau



① : $\varepsilon_0^2 = 0$

② : ε_0^2 weak (~.01)

③ : ε_0^2 strong (~10)

Extensions: (HW)

→ synchronization by quasi-harmonic (narrow band) stochastic force

→ mutual synchronization of noisey oscillators.