Lecture 8-11: Synchronization and Turbulence

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1 Brief Review

In the previous lecture, we discussed the method to convert two harmonic oscillators into a single harmonic oscillator and the problem of the synchronization with noise. Let's consider the phase dynamics of a single harmonic oscillator with additive noise described by the following Langevin equation,

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = w_0 + \epsilon Q(\phi, t) + \mathrm{Noise(t)} \tag{1}$$

By taking $\psi = \phi - wt$, we can rewrite the previous equation as,

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -v + \epsilon q(\psi) + \mathrm{Noise(t)} \tag{2}$$

where $v = w - w_0$ is the mismatch frequency and ϵ is a constant of coupling strength. In this formulation, we can expect that the game is among the mismatch, the coupling and the noise. With the Fokker-Planck method, the resulting beat frequency v.s. mismatch plots show softening effects from this additive noise term. Based on the graph of beat frequency v.s. mismatch in the previous lecture, we can see that the stronger the noise is, the narrower the synchronization region is. This result implies that synchronization in the noisy environment is also determined by the intensity of the noise and the competition game in this dynamics is actually between the coupling and the noise terms. The mismatch loses its significance in synchronization when strong noise is present.

2 Synchronization in Oscillatory Media

Now, let's increase the number of oscillators to N and add some spatial information for them. Each oscillator is self-sustained and sits in some position x_i, $i = 1, ..., N$. Well, this construction looks quite hard to analyze. Extra spatial dynamics always makes many behaviors intractable. Before working on this complicated model, let's take a look at one simple and special case, the nearest neighbor coupling in $D = 1$. For each oscillator k, $k = 1, ..., N$, the dynamics is followed by,

$$\frac{\mathrm{d}\phi_k}{\mathrm{d}t} = w_k + \epsilon q(\phi_{k-1} - \phi_k) + \epsilon q(\phi_{k+1} - \phi_k) \tag{3}$$

where the boundary condition is given by $\phi_0 = \phi_1, \phi_{N+1} = \phi_N$.
If $\epsilon \to 0$, then all oscillators are decoupled and they are just self-sustained individual oscillators. The dynamics is simply determined by each w_k. If $\epsilon \gg 1$, then $\epsilon \gg |w_k|$. All oscillators will be synchronized eventually. In between, it's obvious that there is a competition between the coupling and the natural frequency. What we can expect is the existence of partial synchronizations, in which some ($\ll N$) different frequencies clusters are present. In our nearest neighbor setting, different clusters of synchronized oscillators are expected, shown in the figure below. This dynamics is also observed in the spin glass model.

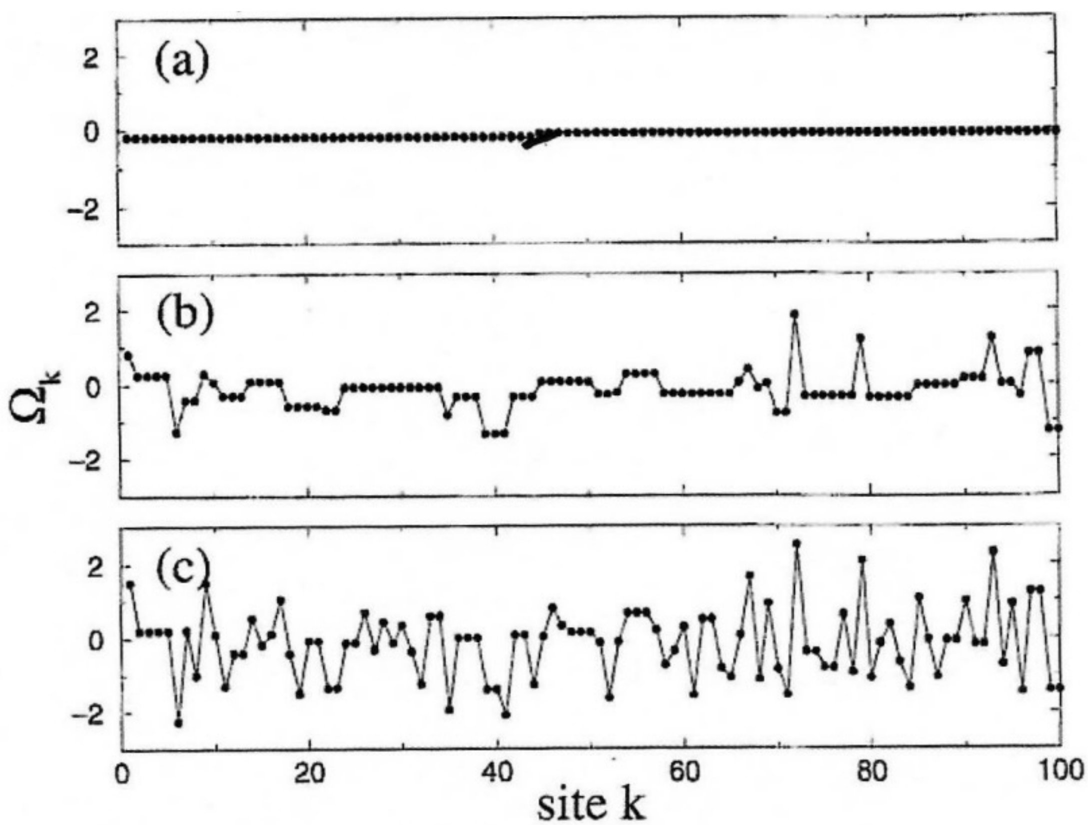


Figure 11.2. Clusters in a lattice (11.1) of 100 phase oscillators with random natural frequencies (normally distributed with unit variance) and the coupling function $q(x) = \sin x$. (a) A two-cluster state at $\varepsilon = 4$. (b) Many relatively large clusters at $\varepsilon = 1$. (c) A few small clusters plus many nonsynchronized oscillators at $\varepsilon = 0.2$.

2.1 Continuum Limit

In the real world applications, the oscillators are generally treated as a continuous oscillatory medium instead of the discrete one. In this section, we will derive the partial differential equation that can describe its evolution, using the continuum limit method. Starting from our favorite toy model (3), we take the spacing between neighboring sites going to zero and the coupling constant going to infinity. Assume q is smooth. Then we are ready to expand it,

$$\frac{\partial \phi_k}{\partial t} = w_k + \epsilon q(\phi_{k-1} - \phi_k) + \epsilon q(\phi_{k+1} - \phi_k) \tag{4}$$

$$\simeq w_k + \epsilon q'[\phi_{k-1} - 2\phi_k + \phi_{k+1}] + \epsilon \frac{q''}{2}[(\phi_{k-1} - \phi_k)^2 + (\phi_{k+1} - \phi_k)^2] + ... \tag{5}$$

By taking $\epsilon \to \frac{\tilde{\epsilon}}{(\Delta x)^2}$ and $\phi_{k+1} - \phi_k = \mathcal{O}(\Delta x)$, we get

$$w_k + \frac{\tilde{\epsilon}}{(\Delta x)^2} q'[\phi_{k-1} - 2\phi_k + \phi_{k+1}] + \frac{\tilde{\epsilon}}{(\Delta x)^2}\frac{q''}{2}[(\phi_{k-1} - \phi_k)^2 + (\phi_{k+1} - \phi_k)^2] + ... \tag{6}$$

The discretization technique tells us that $\nabla^2 \phi = \frac{\phi_{k-1} - 2\phi_k + \phi_{k+1}}{(\Delta x)^2}$ and $\nabla \phi = \frac{\phi_{k-1} - \phi_k}{(\Delta x)}$ as $\Delta x \to 0$. With simple algebra and proper choice of α and β, we get the following Phase Diffusion/Evolution equation

$$\frac{\partial \phi(x,t)}{\partial t} = w(x) + \alpha \nabla^2 \phi + \beta (\nabla \phi)^2 + h.o.t \tag{7}$$

where if $\alpha < 0$, then this will lead to the instability.
Note that this equation contains both Snell terms and KPZ/Burger terms. Let's take $v = \nabla \phi$ and take spatial derivative to b.h.s. of (7), then we have the familiar form,

$$\frac{\partial v}{\partial t} = \nabla w(x) + 2\alpha \nabla^2 v + \beta v (\nabla v) \tag{8}$$

2.2 Simple Case

First, let's consider a 1-D case with non-trivial natural frequency $w(x)$ term. The equation (7) can be simplified by the Hopf-Cole substitution $\phi = \frac{\alpha}{\beta}\ln(U)$,

$$\frac{\alpha}{\beta}\frac{1}{U}\frac{\partial U}{\partial t} = w(x) + \{\alpha \frac{\alpha}{\beta}(\frac{1}{U}\partial_x^2 U - \frac{1}{U^2}(\partial_x U)^2) + \beta \frac{\alpha^2}{\beta^2}\frac{1}{U^2}(\partial_x U)^2\} \tag{9}$$

$$= w(x) + \frac{\alpha^2}{\beta}\frac{1}{U}\partial_x^2 U \tag{10}$$

So,

$$\frac{\partial U}{\partial t} = \frac{\beta}{\alpha} w(x) U + \alpha \partial_x^2 U \tag{11}$$

Take $U(x,t) = U(x)e^{\lambda t}$, then we have the following Sturm-Liouville problem,

$$\lambda U(x) = \frac{\beta}{\alpha} w(x)U(x) + \alpha \partial_x^2 U(x) \tag{12}$$

If w is constant, then we have the plane wave solution for U in Fourier space, $x \to k$. The phase solution can be, thus, calculated as,

$$\phi(x,t) = kx + (w + \beta k^2)t + \phi_0 \tag{13}$$

This solution has several properties:

1. Non-zero phase shift between points. Clearly, position matters in this solution. Different points must have a non-zero phase shift.

2. Competition between the dispersive term and the sign of β in this solution.

3. Sensitive to boundary conditions. If we set our boundary condition to $\nabla\phi = 0$, then we have a unique solution for $k = 0$.

2.3 Phase Roughening and Decoherence

Next, we will discuss how noise affects the synchronization in a large system. From the previous discussion, we can simply have an additive noise term to our phase diffusion equation and assume w, α, β are constant. In this case, equation (7) can be rewritten as KPZ equation,

$$\frac{\partial \phi(x,t)}{\partial t} = w + \alpha \nabla^2 \phi + \beta (\nabla \phi)^2 + \eta(x,t) \tag{14}$$

where $\eta(x,t)$ is the zero-mean white noise, $< \eta(x,t) >= 0, < \eta(x,t)\eta(x',t') >= 2\sigma^2 \delta_{x,x'}\delta(t-t')$.
This equation is well-studied in the theory of roughening interface. An interesting question that we can ask in this equation is how rough does the interface get. Roughening refers to the deviations from the mean value. In our context, the interface is our phase profile, $\phi(x,t)$. It is not hard to see that a "smooth profile" is reached when the whole system synchronizes globally and a "rough profile" appears when the decoherence is happening, where many modest domains or different synchronized clusters are present in the whole system. Then, how to relate the scale, the effective coherence domain size, of the synchronization to the roughness? A spectrum of phases. A direct approach now is to calculate the probability density function of the phase domains using Fokker-Planck method.
Let's first take a simple example and see how this works. Assume the weak fluctuation limit. The nonlinear term vanishes. The Langevin equation (14) becomes,

$$\frac{\partial \phi(x,t)}{\partial t} = w + \alpha \nabla^2 \phi + \eta(x,t) \tag{15}$$

In the Fourier space, it becomes the Brownian Motion,

$$\frac{\partial \phi_k}{\partial t} = w\delta_{k,0} - \alpha k^2 \phi_k + \tilde{\eta}(x,t) \tag{16}$$

where $< \tilde{\eta}_k(t)\tilde{\eta}'_k(t') >= 2\sigma^2 \delta_{k,k'}\delta(t-t')$.
Now, the Fokker-Planck equation is followed from the Langevin equation. Let $P = P(\phi_k)$ be the pdf of the phases for $k \neq 0$ and $D = \sigma^2$. Then we have,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \phi_k}\{(-k^2\alpha\phi_k)P - \frac{\partial}{\partial \phi_k}DP\} \tag{17}$$

The steady-state solution is,

$$P_{ss}(\phi_k) = c\,\exp[\frac{-\phi_k^2}{D/\alpha k^2}] \tag{18}$$

It shows that the pdf of ϕ_k is Gaussian with zero mean and variance, $\mathrm{Var}(\phi_k) = D/\alpha k^2 = \sigma^2/\alpha k^2$, which will diverge at large scale.
For the spectrum,

$$\begin{aligned} |\phi_k(\gamma)| &= \frac{2\sigma^2}{\gamma^2 + (\alpha k^2)^2} && (19)\\ |\phi_k|^2 &= \int \mathrm{d}\gamma \frac{2\sigma^2}{\gamma^2 + (\alpha k^2)^2} && (20)\\ &= \int \frac{\mathrm{d}\gamma}{\alpha k^2}\frac{2\sigma^2}{(\gamma/(\alpha k^2))^2 + 1}\frac{\alpha k^2}{(\alpha k^2)^2} && (21)\\ &\approx 1/\alpha k^2 && (22) \end{aligned}$$

Note that the spectrum of phase fluctuation will also diverge at large scale as well. Therefore we have,

$$\begin{aligned} |\tilde{\phi}|^2 &= \int_{K_{min}}^{K_{max}} \mathrm{d}k k^{d-1}\frac{1}{\alpha k^2} && (23)\\ &= \begin{cases} 1/K_{min} \sim L_{max} \text{ if d} = 1 \\ \ln L_{max} \text{ if d} = 2 \\ C \text{ (infrared convergent intensity), if d} \geq 3 \end{cases} && (24) \end{aligned}$$

This implies that the phase roughness is a strong function of dimensionality of the system. The phase profile is rough in dimension $d = 1, 2$ due to the divergence of L_{max} and there is no roughening in dimensions $d \geq 3$ since the variance of the phase profile is finite for any scale.
This roughening-nonroughening transition can be interpreted as the coherence-decoherence transition. In the 1-D case, all observed frequencies can be the same at all points. Therefore, the oscillations are synchronized, but they're not coherent. The phase profile can

be viewed as a random walk, where the coherence happens in the small scale and the decoherence appears in the large scale.
For the nonlinear case, the nonlinear term, $\frac{(\nabla\phi^2)}{2}$, can be treated as a renormalized perturbation T so that $\alpha_T(|\tilde{\phi}_k|^2, ...) \sim |\phi_k|^2$ and $\frac{\sigma^2}{\alpha} \rightarrow \frac{\sigma^2}{\alpha+\alpha_T}$.

3 Kuramoto Dynamics

In this section, we want to examine the self-organization of oscillator ensembles via phase transition to synchronous behaviors. There are several obvious analogies we can make between the phase transition and the self-organization of oscillator ensembles.

Phase Transition	Oscillator Ensembles
Mean Magentization M	Mean Field Q(phase forcing)
Order	Synchronization
Disorder	Asynchrony
Thermal Fluctuation	Noise
$s_i s_j$ Coupling	Oscillator Coupling

3.1 Kuramoto Transition

Let's consider N mutually coupled oscillators with $N \gg 1$ and different natural frequencies w_k in 0D. The dynamics is therefore described by the following Kuramoto model,

$$\frac{\mathrm{d}\phi_k}{\mathrm{d}t} = w_k + \frac{\epsilon}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_k) \tag{25}$$

Here we suppose the natural frequencies of the oscillators are distributed in some range. In the thermodynamic limit $N \rightarrow \infty$, we can describe the distribution, $g(w)$, to be a Bell-shaped distribution with only one maximum and symmetric around $\bar{w}$, $g(\bar{w} - x) = g(\bar{w} + x)$. The coupling term is set to be proportional to N^{-1}. Why $1/N$? It guarantees that the coupling is finite in the thermodynamic limit. Otherwise, the coupling term will trivially dominate the mismatch term as $N \rightarrow \infty$. Notice that this equation is similar to what we've seen in the previous lecture on the Adler's equation. Similarly, we can also expect the competitions between the coupling strength and the frequency spread(e.x. peak and width) in the dynamics.

3.2 Mean Field Approach

Since in the Kuramoto model all oscillators couple and all coupling strengths are identical, we can expect mean field approach to be accurate as $N \rightarrow \infty$. In the theory of mean field, there are several things needed to be considered.

1. Order Parameter
2. Represent mean field assembly to oscillators.(e.x self-consistency)
3. Represent coupling of each oscillator to mean field environment.

3.2.1 Order Parameter

First, introduce the complex mean field of the population

$$Z = X + iY = Ke^{i\theta} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} \tag{26}$$

The coupling(entrainment) term can be rewritten as,

$$\frac{1}{N}\sum_{i=1}^{N}\sin(\phi_j - \phi_k) = \frac{1}{N}\sum_{i=1}^{N}\sin(\phi_j)\cos(\phi_k) - \frac{1}{N}\sum_{i=1}^{N}\cos(\phi_j)\sin(\phi_k) \tag{27}$$

$$= K\sin(\theta - \phi_k) \tag{28}$$

Therefore, the e.o.m is a system of oscillators forced by the mean field,

$$\frac{\mathrm{d}\phi_k}{\mathrm{d}t} = w_k + \epsilon K \sin(\theta - \phi_k) \tag{29}$$

3.2.2 Mean Field to Oscillators

Our next goal is to determine the synchronous/asynchronous region of oscillators in terms of the mean field.

$$Z = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} \tag{30}$$

From this, we have,

$$K = \int_{-\pi}^{\pi} e^{i\psi} n(\psi)\mathrm{d}\psi \tag{31}$$

where $n(\psi)$ is the phase distribution.
By taking $\theta = \bar{w}t, K = \text{constant}, \psi_k = \phi_k - \bar{w}t$,equation (29) is now rewritten as,

$$\frac{\mathrm{d}\psi_k}{\mathrm{d}t} = w_k - \bar{w} - \epsilon K \sin(\psi_k) \tag{32}$$

So, the oscillator is entrained by mean field if $\psi_k = \sin^{-1}[\frac{w_k - \bar{w}}{\epsilon k}]$, where $|\frac{w_k - \bar{w}}{\epsilon k}| \leq 1$. This is the synchronous solution. Otherwise, we obtain the asynchronous solution, in which phase ψ_k rotates. Note that phase slip will happen when in the asynchronous state if phase lingers near the peak that is close to the entrainment,ϵK.

3.2.3 Reconstruction of Mean Field

The last job is to reconstruct the mean field. From the previous discussion, we have,

$$K = \int_{-\pi}^{\pi} e^{i\psi} n(\psi) \mathrm{d}\psi \tag{33}$$

where the phase distribution is decomposed as $n(\psi) = n_{synch}(\psi) + n_{asynch}(\psi) = n_S(\psi) + n_{AS}(\psi)$.
For $n_S(\psi)$, it is simply the distribution of the natural frequencies,

$$n_S(\psi) = g(\psi) = g(w) |\frac{\mathrm{d}w}{\mathrm{d}\psi}| \tag{34}$$

From the previous definition, these are $\frac{\mathrm{d}w}{\mathrm{d}\psi} = \epsilon K \cos(\psi)$ and $w = \bar{w} + \epsilon K \sin(\psi)$. Therefore,

$$n_S(\psi) = g(\bar{w} + \epsilon K \sin(\psi)) \epsilon K \cos(\psi) \tag{35}$$

For $n_{AS}(\psi)$, this is the distribution of the relative amount of time that oscillators spend at each value of ψ in the total period. Namely,

$$n_{AS}(\psi) = t_\psi |T_\psi|^{-1} = |\dot{\psi}|^{-1} |T_\psi|^{-1} \tag{36}$$

Note that,

$$\dot{\psi} = w - \bar{w} - \epsilon K \sin(\psi) \tag{37}$$

$$T_\psi = \int_0^{2\pi} \mathrm{d}\psi / |w - \bar{w} - \epsilon K \sin(\psi)| \tag{38}$$

$$\approx 1/[(w - \bar{w})^2 - \epsilon^2 k^2]^{\frac{1}{2}} \tag{39}$$

So the probability of observing at ψ with w is,

$$P(\psi, w) = \frac{1}{2\pi} \frac{[(w - \bar{w})^2 - \epsilon^2 k^2]^{\frac{1}{2}}}{|w - \bar{w} - \epsilon K \sin(\psi)|} \tag{40}$$

Together with distribution of frequency over the asynchronous region,

$$n_{AS}(\psi) = \int_{|w - \bar{w}| > \epsilon K} \mathrm{d}w g(w) P(\psi, w) \tag{41}$$

$$= \int_{\bar{w} + \epsilon K}^{\infty} \mathrm{d}w g(w) P(\psi, w) + \int_{-\infty}^{\bar{w} - \epsilon K} \mathrm{d}w g(w) P(\psi, w) \tag{42}$$

$$= \int_{\epsilon K}^{\infty} \mathrm{d}x x g(\bar{w} + x) \frac{[x^2 - \epsilon^2 k^2]^{\frac{1}{2}}}{x^2 - \epsilon^2 K^2 \sin^2(\psi)} \tag{43}$$

Finally,

$$K = \int_{-\pi}^{\pi} e^{i\psi}(n_S(\psi) + n_{AS}(\psi))\mathrm{d}\psi \tag{44}$$

Note that n_{AS} has period of π in ψ. This is saying that this term has no contribution to the integral. Therefore,

$$K = \int_{-\pi}^{\pi} e^{i\psi} n_S(\psi)\mathrm{d}\psi \tag{45}$$

$$= \int_{-\pi}^{\pi} g(\bar{w} + \epsilon K \sin(\psi))\epsilon K \cos(\psi) e^{i\psi}\mathrm{d}\psi \tag{46}$$

So, this can be expanded into real and imaginary parts,

$$K = \int_{-\pi/2}^{\pi/2} g(\bar{w} + \epsilon K \sin(\psi))\epsilon K \cos^2(\psi)\mathrm{d}\psi \tag{47}$$

$$0 = \int_{-\pi/2}^{\pi/2} g(\bar{w} + \epsilon K \sin(\psi))\epsilon K \cos(\psi)\sin(\psi)\mathrm{d}\psi \tag{48}$$

$$= \int_{-\epsilon K}^{\epsilon K} \mathrm{d}x \frac{x}{\epsilon K} g(\bar{w} + x) \tag{49}$$

where equation (46) is the self-consistency condition that determines the strength of the mean field K and equation (47),(48) determines the distribution of frequencies.
A clever guess to the analytical solution of (48) is the Lorentzian distribution,

$$g(x) = \frac{\gamma}{\pi[(w - \bar{w})^2 + \gamma^2]} \tag{50}$$

where $\bar{w}$ is the peak and γ is the width.
The competitions between the coupling strength ϵ and the range of frequencies γ are described by the solution of equation (46),

$$K = \sqrt{1 - \frac{2\gamma}{\epsilon}} \tag{51}$$

From this, it's not hard to see that $\epsilon_c = 2\gamma$ for the onset of the synchronization. Therefore, $K \sim (\epsilon - \epsilon_c)^{\frac{1}{2}}$. Assume we don't know the existence of Lorentzian distribution and treat g as a unimodal distribution. Then, for small K, we can Taylor expand $g(\bar{w} + \epsilon K \sin(\psi))$

$$g(\bar{w} + \epsilon K \sin(\psi)) \approx g(\bar{w}) + \frac{g''}{2}\epsilon^2 K^2 \sin^2(\psi) \tag{52}$$

Put this into (46), we get $\epsilon_c = \frac{2}{\pi g(\bar{w})}$ and $K^2 \approx \frac{8g(\bar{w})}{|g''|\epsilon^3}(\epsilon - \epsilon_c)$.

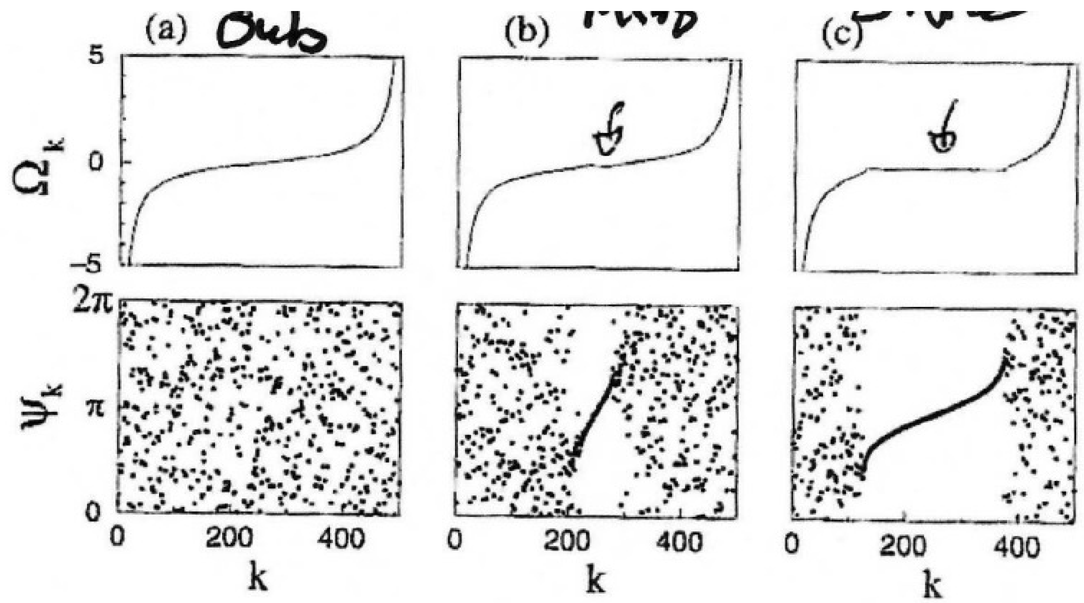


Figure 12.1. Dynamics of a population of 500 phase oscillators governed by Eq. (12.1). The distribution of natural frequencies is the Lorentzian one (12.13) with $\gamma = 0.5$ and $\bar{\omega} = 0$; the critical value of coupling is $\varepsilon_c = 1$. (a) Subcritical coupling $\varepsilon = 0.7$. The oscillators are not synchronized, the mean field fluctuates (due to finite-size effects) around zero. (b) Nearly critical situation $\varepsilon = 1.01$. A very small part of the population near the central frequency is synchronized. The observed frequencies $\Omega_k = \langle \dot{\phi}_k \rangle$ are the same for these entrained oscillators. (c) $\varepsilon = 1.2$, a large part of the population is synchronized, the mean field is large. The amplitude of the mean field is $K \approx 0.1$ for (b) and $K \approx 0.41$ for (c), in good agreement with the formula (12.14).

3.3 More on Kuramoto Transition

There are more points worth discussing about the Kuramoto model.

1. Is the Kuramoto transition a "true" bifurcation? For instance, take $\mu = (K - K_c)/K$. Does $K = 0$ state goes unstable when $\mu > 0$.
   It's not yet clear. Since the number of phase configurations for macro state of particular K is infinite, it is tough to show rigorously.

2. What fraction of oscillators participate in a synchronized cluster?
   Recall that only synchronized population is participating. That is,

$$\psi_k = \bar{w}t + \sin^{-1}[\frac{w_k - \bar{w}}{\epsilon K}] \tag{53}$$

So, the fraction of synchronized oscillators is,

$$r = \frac{N_S}{N} = \int_{\bar{w}-\epsilon K}^{\bar{w}+\epsilon K} \mathrm{d}w g(w) \tag{54}$$

Near the threshold,

$$r \approx 2\epsilon K g(\bar{w}) \tag{55}$$

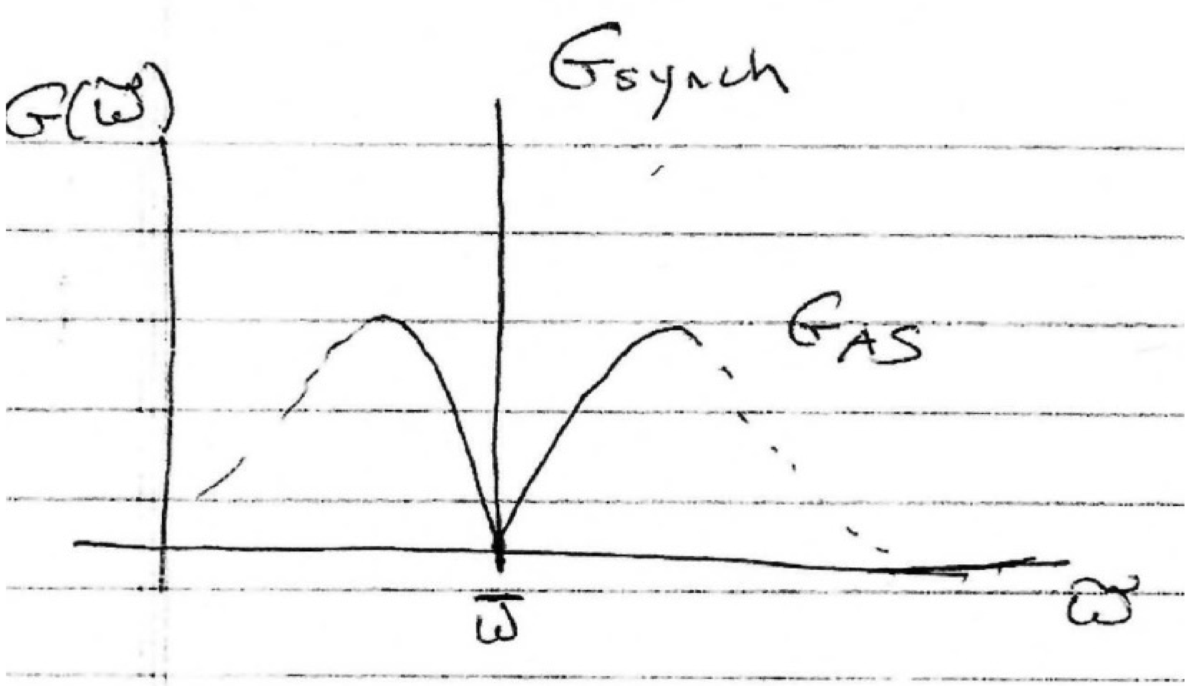
$$= 2\epsilon[\frac{8g(\bar{w})}{|g''|\epsilon^3}(\epsilon - \epsilon_c)]^{\frac{1}{2}} g(\bar{w}) \tag{56}$$

3. What is the observable distribution of frequencies?
Note that it's important to distinguish the input distribution of frequencies, $g(w)$, and the distribution of effective frequencies,$G(\tilde{w})$, that an experimentalist would actually measure.
Now, use the same analysis technique that we did on computing the phase distribution. Then we get,

$$G(\tilde{w}) = G_S(\tilde{w}) + G_{AS}(\tilde{w}) \tag{57}$$

$$G_S(\tilde{w}) = r\delta(\tilde{w} - \bar{w}) \tag{58}$$

$$G_{AS}(\tilde{w}) = g(\bar{w} + [(\tilde{w} - \bar{w})^2 + (\epsilon K)^2]^{\frac{1}{2}}) \frac{|\tilde{w} - \bar{w}|}{((\tilde{w} - \bar{w})^2 + (\epsilon K)^2)^{\frac{1}{2}}} \tag{59}$$



From the figure above, $g(\bar{w})$, a unimodal distribution, and $G(\tilde{w})$ are fundamentally different. In $G(\tilde{w})$, the frequencies near the central peak ,$\bar{w}$, are pulled to peak by the Kuramoto transition and the central peak naturally depletes the neighboring population of non-synchronized oscillators.

4. What should we expect when the spatial dimension is present? N-D v.s. 0-D. N-D problem is certainly of our interests. Especially in neuroscience, a detailed and complete analysis in 3-D will be highly-demanded. However, the problem of the range of coupling v.s. the scale of system naturally occurs and this makes the analysis much more complicated.

4 Kuramoto Dynamics with Noise

Now, we're interested in the synchronization in the presence of noise. As we did in section 2.3, we put another additive white noise term in the Kuramoto model. The phase dynamics is described by the Langevin equation, for $k = 1, ..., N$,

$$\frac{\mathrm{d}\phi_k}{\mathrm{d}t} = w_0 + \frac{\epsilon}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_k) + \eta_k(t) \tag{60}$$

where $<\eta(t)>= 0, <\eta_m(t)\eta_n(t')>= 2\sigma^2\delta_{n,m}\delta(t-t')$ and assume the natural frequencies for all oscillators are identical. As what we've discussed previously, the competition for this stochastic model is between the coupling and the noise instead of the dispersion and the coupling.

4.1 Solve the Noisy Kuramoto Model

Take $\psi_k = \phi_k - w_0 t$ and define the mean field in the thermodynamic limit $N \to \infty$, $Z = X + iY = \frac{1}{N}\sum_{j=1}^{N} e^{i\psi_j} \to \int_0^{2\pi}\mathrm{d}\psi e^{i\psi}\rho(\psi,t)$, where $\rho(\psi)$ is the pdf of ψ_k. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}\psi_k}{\mathrm{d}t} &= \frac{\epsilon}{N}\sum_{j=1}^{N}\sin(\psi_j - \psi_k) + \eta_k(t) && (61)\\
&= \frac{-i\epsilon}{2N}[\sum_{j=1}^{N} e^{i(\psi_j-\psi_k)} - \sum_{j=1}^{N} e^{-i(\psi_j-\psi_k)}] + \eta_k(t) && (62)\\
&= \frac{-i\epsilon}{2N}[\sum_{j=1}^{N} e^{i\psi_j}e^{-i\psi_k} - \sum_{j=1}^{N} e^{-i\psi_j}e^{i\psi_k}] + \eta_k(t) && (63)\\
&= \frac{-i\epsilon}{2}(Ze^{-i\psi_k} - Z^* e^{i\psi_k}) + \eta_k(t) && (64)\\
&= \epsilon(-X\sin(\psi_k) + Y\cos(\psi_k)) + \eta_k(t) && (65)
\end{aligned}$$

The Fokker-Planck equation immediately follows,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \psi}[< \frac{\mathrm{d}\psi}{\mathrm{d}t} > \rho - \frac{\partial}{\partial \psi}(\frac{< \eta(t)\eta(t+\Delta t) >}{2\Delta t}\rho)] \tag{66}$$

$$= \frac{\partial}{\partial \psi}[\epsilon(X\sin(\psi) - Y\cos(\psi))\rho + \sigma^2\frac{\partial}{\partial \psi}\rho] \tag{67}$$

Observe that this FP-equation is nonlinear. In order to solve this equation, we need to use the hierarchy trick. Expand the density into Fourier modes,

$$\rho(\psi, t) = \frac{1}{2\pi}\sum_l \rho_l(t)e^{il\psi} \tag{68}$$

The normalization of zero mode naturally leads to $\rho_0 = 1$.

$$Z = \int_0^{2\pi} \mathrm{d}\psi e^{i\psi}\rho(\psi, t) = \rho_1^* = \rho_{-1} \tag{69}$$

Now, the nonlinear FP-equation couples harmonics,

$$\frac{\partial \rho_l}{\partial t} = \frac{l\epsilon}{2}[\rho_{l-1}\rho_1 - \rho_{l+1}\rho_{-1}] - \sigma^2 l^2 \rho_l \tag{70}$$

where the first term is the drifting with death-birth process(becomes mean field when $l = 1$) and the second term is the diffusion. So,

$$\dot{\rho_1} = \frac{\epsilon}{2}[\rho_1 - \rho_2\rho_{-1}] - \sigma^2\rho_1 \tag{71}$$

$$\dot{\rho_2} = \epsilon[\rho_1^2 - \rho_3\rho_{-1}] - 4\sigma^2\rho_2 \tag{72}$$

$$\dot{\rho_3} = \frac{3\epsilon}{2}[\rho_1\rho_2 - \rho_4\rho_{-1}] - 9\sigma^2\rho_3 \tag{73}$$

$$\ldots \tag{74}$$

It's the hierarchy of coupled modes amplitude equation.
Observe that the trivial solution of this system, in which $\rho_0 = 1$ and all other Fourier modes vanish $\rho_l = 0$ for $l \in \mathbb{Z}^*$, is the homogeneous distribution of phases. For the non-trivial one, let's linearize (70) as,

$$\dot{\rho_1} \approx (\frac{\epsilon}{2} - \sigma^2)\rho_1 \tag{75}$$

It's clear that ρ_1 is unstable if $\epsilon > 2\sigma^2$ and stable if $\epsilon < 2\sigma^2$. Therefore, we get the critical value $\epsilon_c = 2\sigma^2$.
Assume $\epsilon \sim 2\sigma^2$ is near the threshold, then,

$$\frac{\mathrm{d}\dot{\rho_1}}{\mathrm{d}\rho_1} \sim (\frac{\epsilon}{2} - \sigma^2) \tag{76}$$

$$\frac{\mathrm{d}\dot{\rho_2}}{\mathrm{d}\rho_2} \sim -4\sigma^2 \tag{77}$$

This implies that $l = 2$ decays rapidly compared to $l = 1$. Therefore,

$$\frac{\mathrm{d}\rho_2}{\mathrm{d}t} = \epsilon(\rho_1^2 - \rho_3\rho_1^*) - 4\sigma^2\rho_2 \approx 0 \tag{78}$$

$$\rho_2 = \frac{\epsilon}{4\sigma^2}\rho_1^2 \tag{79}$$

Then, we plug this into (70),

$$\dot{\rho}_1 = \frac{\epsilon}{2}[\rho_1 - \frac{\epsilon}{4\sigma^2}\rho_1^2\rho_{-1}] - \sigma^2\rho_1 \tag{80}$$

$$= (\frac{\epsilon}{2} - \sigma^2)\rho_1 - \frac{\epsilon^2}{8\sigma^2}|\rho_1|^2\rho_1 \tag{81}$$

Again, we obtain the Landau-Stuart equation, where the first term is the driving force and the second is the saturation. Finally, we can calculate the steady state solution $|Z|^2 = |\phi_1|^2$ near the threshold,

$$|Z|^2 = (\epsilon - 2\sigma^2)\frac{4\sigma^2}{\epsilon^2} \tag{82}$$

$$= \frac{\epsilon - \epsilon_c}{\epsilon}\frac{4\sigma^2}{\epsilon} \tag{83}$$

That is $|Z| = |\phi_1| \sim (\epsilon - \epsilon_c)^{\frac{1}{2}}$. The mean field grows at the transition as a square root of $(\epsilon - \epsilon_c)$. This also illustrates the similarity between the population of noisy oscillators and the mean field theory of phase transition. Also, we notice that $\rho \approx 1+$.
Several other interesting questions can be raised on this stochastic model.

1. How much "noise" does it take to "randomize" phase?(e.x. transform the pdf of ϕ to Gaussian)

2. What is the effects of quenching disorder in couplings on the pdf of phase?

4.2 Extensions and Generalizations

In the previous discussions, we analyzed two models:1.Uniform coupling Kuramoto transition with a symmetric unimodal distribution $g(w)$;2. Uniform coupling Kuramoto transition with constant frequency $w = w_0$ and additive white noise. Many other variations are also of our interests. They can be built based on the above two types.

1. A mixture of both. An immediate model is the combination of both types, Kuramoto model with distribution of natural frequencies and noise. Namely, the FP-equation will look like,

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial\psi}[(\text{frequency mismatch and coupling})\rho - \frac{\partial}{\partial\psi}(\frac{< \eta(t)\eta(t+\Delta t) >}{2\Delta t}\rho)] \tag{84}$$

2. Generalized attractive coupling. In the Kuramoto model, the coupling term is the simplest sine wave. Beyond the sine wave, we can consider other nonlinear couplings or even the generalized nonlinear coupling $q(...)$. Then repeat the analysis on order parameter and transition behaviors.

3. Random Coupling. Instead of the fixed coupling strength, a distribution of coupling strength can be considered. For instance,

$$\frac{\mathrm{d}\psi_k}{\mathrm{d}t} = w_k + \sum_{j=1}^{N} \frac{\epsilon_{j,k}}{N} \sin(\psi_j - \psi_k + \eta_{j,k}) \tag{85}$$

where $\epsilon_{j,k} > 0$ is sampled from $pdf(\epsilon)$ and $\eta_{j,k}$ can be either fixed or randomly sampled from some pdf of η so that it can produce frustrations to the phases.

4. Hysteretic Transition. Namely, put some inertia to the phase. Therefore,

$$I\frac{\mathrm{d}^2\psi_k}{\mathrm{d}t} + \frac{\mathrm{d}\psi_k}{\mathrm{d}t} = w_k + \frac{\epsilon}{N}\sum_{j=1}^{N} \sin(\psi_j - \psi_k) + \eta_k(t) \tag{86}$$

The inertia effect makes the whole system history-dependent. People in engineering and science always show interests in this kind of problem. Even the Google translator uses some variations of it in language translations.

5 Phase Turbulence

So far, we've worked on the derivation of the phase diffusion equation and analyzed the equation in 1-D. Now we proceed to the turbulence.
What's turbulence? Technically, turbulence means that the system has at least 1 positive Lyapunov exponent. Many different types of turbulence are observed in the dynamics. For instance, amplitude/defect turbulence and turbulence solutions with coherent structures.

5.1 Complex Ginzburg Landau Model

Let's back to the complex Ginzburg-Landau model,

$$\partial_t A = A - (1 - i\alpha)|A|^2 A + (1 + i\beta)\nabla^2 A \tag{87}$$

where α is the nonlinear frequency shift and the β is the dispersion.
The simplest solution to this equation is the plane wave solution. By taking the general form $A = Re^{i\phi}$, we get

$$iR\partial_t\phi + \partial_t R = R - R^3 - i\alpha R^3 + (1 + i\beta)(iR\nabla^2\phi - R(\nabla\phi)^2 + 2i\nabla\phi\nabla R + \nabla^2 R) \tag{88}$$

By separating the real part and the imaginary part, we obtain the amplitude and phase dynamics,

$$\partial_t R = R - R^3 - 2\beta\nabla\phi\nabla R - \beta R\nabla^2\phi - R(\nabla\phi)^2 + \nabla^2 R \tag{89}$$

$$R\partial_t\phi = -\alpha R^3 + R\nabla^2\phi - \beta R(\nabla\phi)^2 + 2\nabla\phi\nabla R + \beta\nabla^2 R \tag{90}$$

A good approach to analyze complicated is to have a clever choice of the "possible" solution. With this idea in mind, we pick $A = (1+\rho)e^{i(\phi(x,t)-\alpha t)}$. Notice that it is a perturbed solution up to normalization with $R = 1+\rho, \Phi = \phi - \alpha t$. Well, this looks very promising. Put A into (88),(89), we get,

$$\partial_t(1+\rho) = (1+\rho) - (1+\rho)^3 - \beta(1+\rho)\nabla^2\phi - 2\beta\nabla\rho\nabla\phi + \nabla^2(1+\rho) - (1+\rho)(\nabla\phi)^2 \tag{91}$$

$$(1+\rho)\partial_t\phi = -\alpha(1+\rho)^3 + (1+\alpha)\nabla^2\phi + 2\nabla\rho\nabla\phi - \beta(1+\rho)(\nabla\phi)^2 + \beta\nabla^2\rho \tag{92}$$

Then, by linearizing, we have two dynamics,

$$\partial_t\rho \approx -2\rho + \nabla^2\rho - \beta\nabla^2\phi \tag{93}$$

$$\partial_t\phi \approx \nabla^2\phi + \beta\nabla^2\rho - 2\alpha\rho \tag{94}$$

In Fourier space, with $t \to w$ and $\nabla \to k$, we get,

$$-iw\rho \approx -3\rho - k^2\rho + \beta k^2\phi \tag{95}$$

$$-iw\phi \approx -k^2\phi - \beta k^2\rho - 2\alpha\rho \tag{96}$$

By solving (95) and (96),

$$(-iw + 2 + k^2)(-iw + k^2) = \beta k^2(-2\alpha - \beta k^2) \tag{97}$$

Take $\gamma = -iw$,

$$\gamma = -(1+k^2) + \sqrt{1 - 2\alpha\beta k^2 - \beta^2 k^4} \tag{98}$$

Let's study its dynamics. First, expand it in the long wavelength limit, $k \ll 1$,

$$\gamma \approx -(1+\alpha\beta)k^2 - \frac{\beta^2}{2}k^4(1+\alpha^2) + ... \tag{99}$$

In order to have negative Lyapunov exponent, we need to convince ourselves that it's safe to assume that generally the leading term needs to be negative. Then we naturally obtain the Benjamin-Feir-Newell stability criterion $1 + \alpha\beta > 0$, which is a special case of the Eckhaus instability criterion(this will be discussed later in the lecture).
Note that equation (94) tells us,

$$(\gamma + 2 + k^2)\rho = \beta k^2\phi \tag{100}$$

$$|\rho| \approx \frac{\beta k^2}{\gamma + 2 + k^2}|\phi| \tag{101}$$

$$= \frac{\beta}{(\gamma+2)/k^2 + 1}|\phi| \tag{102}$$

Therefore, in the long wavelength limit $k \ll 1$, $|\rho| \ll |\phi|$. This is saying that fluctuations in phase are much larger than fluctuations in amplitude. The critical point for phase instability can be derived by setting $\gamma = 0$,

$$k_c \approx \sqrt{\frac{2|1+\alpha\beta|}{\beta^2(1+\alpha^2)}} \tag{103}$$

The following figures are simulated for the two cases, where one obeys the B.F. condition and the other violates. As we can see visually, in Figure 1 the plane wave solution shows stable solutions, whereas Figure 2 gives phase turbulence. The behavior of this turbulence region is of course interesting to study, but, unfortunately, it is beyond the scope for this lecture notes.

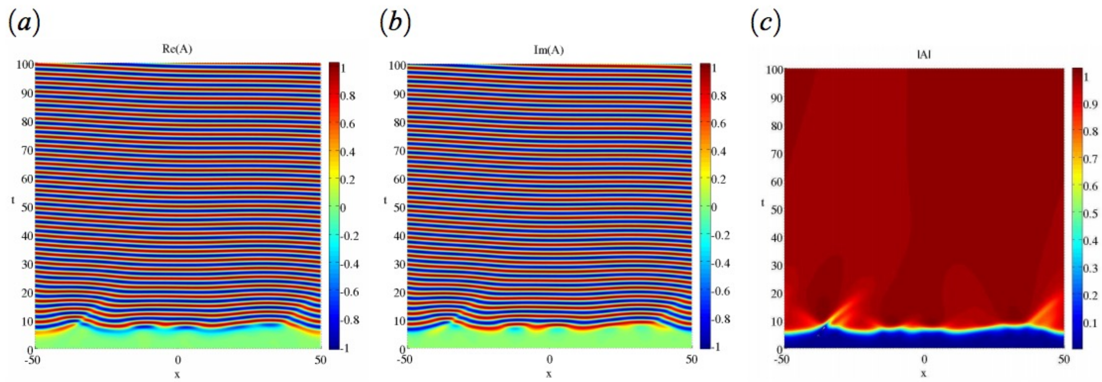


Figure 1: Obey B.F. with $(\alpha, \beta) = (1, 2)$

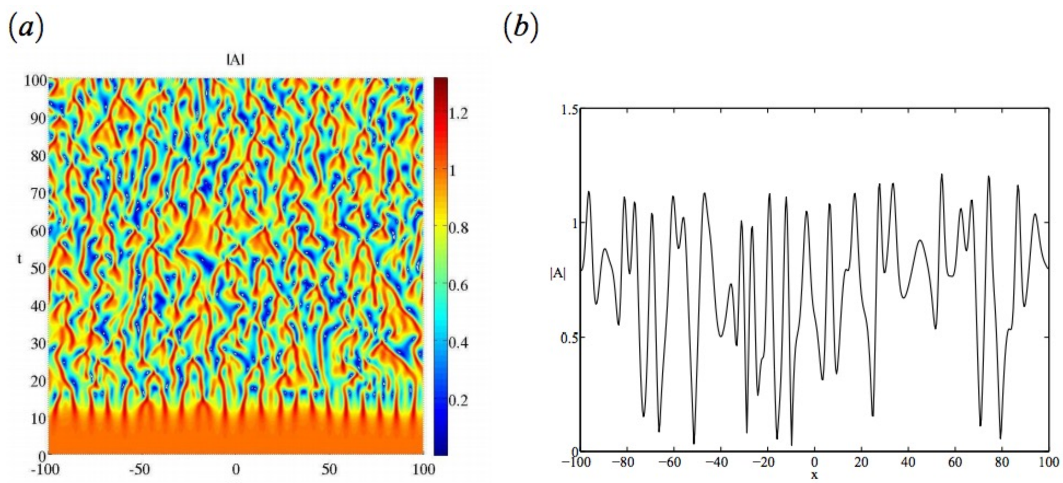


Figure 2: Violate B.F.I with $(\alpha, \beta) = (2, -2)$

5.2 Kuramoto-Sivashinsky Equation

What happens if the Benjamin-Feir-Newell criterion is violated? As the figure shown above, we reach the phase turbulence region. From our previous experience, the non-linear terms shown in the CGL equation could really mess up the calculations. We need a tool or a simple governing equation to describe the dynamics in that region. Here the Kuramoto-Sivashinsky phase equation comes to help. Recall, in the previous discussion, we have,

$$\partial_t \rho = (1+\rho) - (1+\rho)^3 - \beta(1+\rho)\nabla^2\phi - 2\beta\nabla\rho\nabla\phi + \nabla^2(1+\rho) - (1+\rho)(\nabla\phi)^2 \tag{104}$$

$$= -2\rho - 3\rho^2 - \rho^3 + \nabla^2\rho - \beta(1+\rho)\nabla^2\phi - 2\beta\nabla\rho\nabla\phi - (1+\rho)(\nabla\phi)^2 \tag{105}$$

$$(1+\rho)\partial_t\phi = -\alpha(1+\rho)^3 + (1+\alpha)\nabla^2\phi + 2\nabla\rho\nabla\phi - \beta(1+\rho)(\nabla\phi)^2 + \beta\nabla^2\rho \tag{106}$$

In the long wavelength limit, assume $k \sim \mathcal{O}(\epsilon)$ is sufficiently small. With this and (100), we get $\frac{|\rho|}{|\phi|} \sim k^2 \sim \mathcal{O}(\epsilon^2), |\phi| \sim \mathcal{O}(1), |\rho| \sim \mathcal{O}(\epsilon^2)$. So,by approximating up to $\mathcal{O}(\epsilon)$, the amplitude equation becomes,

$$0 \approx -2\rho - \beta\nabla^2\phi - (\nabla\phi)^2 \tag{107}$$

$$\rho \approx -\frac{\beta}{2}\nabla^2\phi - \frac{1}{2}(\nabla\phi)^2 \tag{108}$$

Put the solution into the phase equation and approximate it. The phase equation can be therefore rewritten as,

$$\partial_t\phi = -2\alpha\rho + \nabla^2\phi - \beta(\nabla\phi)^2 \tag{109}$$

$$= -2\alpha[-\frac{\beta}{2}\nabla^2\phi - \frac{1}{2}(\nabla\phi)^2] + \nabla^2\phi - \beta(\nabla\phi)^2 \tag{110}$$

Therefore, the final phase diffusion equation is obtained,

$$\partial_t\phi = (1+\alpha\beta)\nabla^2\phi + (\alpha-\beta)(\nabla\phi)^2 \tag{111}$$

Since we're interested in the turbulence region, $1+\alpha\beta < 0$, we need higher order terms to help us. With some algebra on higher order terms, K-S equation follows,

$$\partial_t\phi = (1+\alpha\beta)\nabla^2\phi + (\alpha-\beta)(\nabla\phi)^2 - M\nabla^4\phi \tag{112}$$

where $M = \frac{\beta^2}{2}(1+\alpha^2)$. Note that in this turbulent region we have a negative diffusion term in the equation and some clustering pattern can be naturally expected from the turbulence.

The K-S equation is another interesting equation of KPZ/Burger family, but it has more interesting behaviors due to the second and fourth order terms. Other than technical things, realistically, KS equation is carrying papers and treasures. A number of different interesting problems are shown to have KS form, cellular pattern problems, reaction-diffusion problems, zigzag instabilities in convective patterns and so on. Interested readers are encouraged to read other technical reviews on KS equations.