Phys 221A Lecture Notes - Lyapunov Exponents and their Relation to Entropy

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Introduction

Previously we discussed the key traits of a chaotic systems, with the simplest description being a dynamical system that exhibits exponential divergence of particle trajectories in phase space. In addition, we characterized some of the key features of chaotic systems such as the appearance of strange attractors and their fractal nature. In order to quantitatively describe chaotic systems, we introduced a geometric construct called the Box Counting dimension which provided a simple yet effective method for quantifying the fractal dimension of a strange attractor in simple systems, with our canonical example being the Bakers Map. The Baker's Map provided a very tractable and elegant example that shows that, counter intuitively, exponential divergence of particle trajectories is possible for a bounded system. With this exponential divergence clearly being made possible by the stretching and folding of phase space which is common to many chaotic systems. The previously discussed Box Counting dimension provided a geometric way of describing the chaotic system and which we expanded in a natural way to the more generalized measure, μ, and generalized dimension, D_q. We then introduced the information dimension D_1 which characterized the rate at which information is produced. We now move away from these geometric descriptions of chaotic systems to a dynamical discussion that utilizes Lyapunov exponents and entropy. In doing so we will also find that there are links between the geometric and the dynamics descriptions.

Lyapunov Exponents

The Lyapunov exponent is a simple way to characterize the dynamics of a chaotic system by looking at the effective degrees of freedom of the system. We begin our discussion of Lyapunov exponents by examining simple one-dimensional maps. Consider a general 1D map given by

$$x_{p+1} = f(x_p)$$

$$x \in [0,1]$$

where f is a function that maps x_p onto x_{p+1}. Without loss of generality we assume that there is a periodic orbit of period p such that for a given starting point x_0 we have

$$x_p = x_0 \; with \; x_p = f^p(x_0)$$

where the map f iterates the points through the orbit defined by $(x_0, x_1, ..., x_p)$. We want to explore the stability of this orbit, therefore we define a small deviation from the orbit, δ_0, and Taylor expand to first order.

$$x_p + \delta_p = f^p(x_0 + \delta_0)$$

$$x_p + \delta_p = f^p(x_0) + \left.\frac{\partial f^p(x)}{\partial x}\right|_{x=x_0} \delta_0$$

$$\Rightarrow \delta_p = \left.\frac{\partial f^p(x)}{\partial x}\right|_{x=x_0} \delta_0$$

we can expand the partial derivative using the chain rule to get

$$\left.\frac{\partial f^p(x)}{\partial x}\right|_{x=x_0} = \frac{\partial f^p(x)}{\partial f^{p-1}(x_{p-1})} \frac{\partial f^{p-1}(x_{p-1})}{\partial f^{p-2}(x_{p-2})} ... \frac{\partial f^{(1)}(x)}{\partial x}$$

$$\Rightarrow \frac{\delta_p}{\delta_0} = \frac{\partial f^p(x_p)}{\partial f^{p-1}(x_{p-1})} \frac{\partial f^{p-1}(x_{p-1})}{\partial f^{p-2}(x_{p-2})} ... \frac{\partial f^{(1)}(x)}{\partial x}$$

$$\Rightarrow \frac{\delta_p}{\delta_0} = \lim_{p\to\infty} \frac{\partial f^p(x_p)}{\partial x} \frac{\partial f^{p-1}(x_{p-1})}{\partial x} ... \frac{\partial f^{(1)}(x)}{\partial x}$$

we then let $\lambda = \frac{\partial f}{\partial x}$ which allows us to write $\delta_p = \lambda^p \delta_0$. Thus we have that

$$\lambda^p = \lim_{p\to\infty} \frac{\partial f^p(x_p)}{\partial x} \frac{\partial f^{p-1}(x_{p-1})}{\partial x} ... \frac{\partial f^{(1)}(x)}{\partial x}$$

$$h(x) = ln(\lambda) = \lim_{p\to\infty} \frac{1}{p} \sum_i ln\left|\frac{\partial f^p}{\partial x}\right|$$

where $h(x)$ is the Lyapunov exponent at x. We also define a related term called the Lyapunov number, $L(x)$, to be

$$L(x) = \lim_{p\to\infty} \left(|f'(x_0)|f'(x_1)|...|f'(x_n)| \right)^{\frac{1}{p}}$$

For $h > 0$ we say that the orbit is *unstable* and deviation will grow exponentially as the number of iterations increases, hence the system will exhibit exponential divergence of particle trajectories. For h=0, as in the case of Hamiltonian systems, we say that the orbit is *superstable* meaning that the deviation from the orbit will remain fixed for all iterations. For the final case where $h < 0$, the orbit is *stable* and any deviation from the orbit will go to zero as the iterations increase. To see why this is so, we return to our equation relating δ_p to δ_0, we have

$$\delta_p = \lambda^p \delta_0$$

$$\delta_p = e^{pln(\lambda)} \delta_0$$

Since $p > 0$, the growth or decay of the deviation depends on the sign of $ln(\lambda)$, i.e. the sign of h.

As an example of the calculation of a Lyapunov exponent for a 1D map consider the $2x$ mod 1 map which is given by

$$x_{p+1} = 2x_p \ mod \ 1$$

$$x \in [0, 1]$$

Using our previous notation for the general 1D map, the mapping function is given by $f(x_p) = 2x_p \ mod \ 1$ and therefore the derivative with respect to x_p is given by $f'(x_p) = 2$. We note that the derivative is not defined at $x = \frac{1}{2}$ due to the discontinuity in the mapping at this point. With this we can then calculate the Lyapunov exponent:

$$h = \lim_{p\to\infty} \frac{1}{p} \sum_i ln\left|\frac{\partial f^p}{\partial x}\right|$$

$$h = \lim_{p\to\infty} \frac{1}{p} \sum_i ln|2|$$

$$h = \lim_{p\to\infty} \frac{pln|2|}{p}$$

$$h = ln|2|$$

We see that the Lyapunov exponent is positive and therefore the system is chaotic. As another example, consider the tent map which is given by

$$x_{p+1} = \begin{cases} 2x_p, & x_p \leq \frac{1}{2} \\ 2(1 - x_p), & x_p > \frac{1}{2} \end{cases}$$

$$x \in [0, 1]$$

For this map, due to the absolute value in the calculation of the Lyapunov exponent, we have that $f'(x_p) = 2$ for both $x_p \leq \frac{1}{2}$ and for $x_p > \frac{1}{2}$. Therefore the Lyapunov exponent for the tent map is the same as the Lyapunov exponent for the $2x$ mod 1 map, that is $h = ln|2|$, thus the tent map exhibits chaotic behavior as well.

Before we proceed to the generalization of the Lyapunov exponent to higher dimensions, it is important to note that the Lyapunov exponent is a local definition of chaos since it is defined independently at each point in the phase space. Therefore a given map can have regions of chaotic motion and regions of stability. Our previous two examples had the same Lyapunov exponent for all points in the 1D phase space. Many systems, especially higher dimensional systems, exhibit regions of chaotic motion and regions of stable periodic motion. Some of the best examples of this are Hamiltonian systems that are slightly perturbed. In Hamiltonian systems, particle orbits are constrained to follow the surface of a resonant tori defined by the energy of the particle. When a slight perturbation is made to the system most of the resonant tori survive and particles continue to exhibit periodic motion along the surface of the tori. As the strength of the perturbation increases, the tori become more and more distorted and eventually overlap leading to the destruction of the tori and chaotic motion as the particles are no longer constrained to a single phase space surface (torus) but are capable or wandering far from their original orbits. However, despite most of the tori being destroyed, some will remain intact, meaning a particle orbit along the surface of one of these surviving tori will remain on the surface of that torus. Therefore we have the mental picture of a phase space that has some regions of fixed, stable periodic orbits and some regions with chaotic motion thus exemplifying the the local nature of the Lyapunov exponent.

The previous discussion leads naturally into a complication that arises when moving to higher dimensions. Not only do we need to consider the position in phase space at which to evaluate the Lyapunov exponent but, in higher dimensions, we also need to consider the *direction* in which we evaluate the Lyapunov exponent. This can lead to the situation where, at a given point in phase space, the Lyapunov exponent may be positive in one direction and negative in the other direction. This corresponds to a stretching along one dimension and squeezing along the other dimension with the overall dynamics of the particle trajectories determined by the competition of these stretching and squeezing components.

With this in mind, we now discuss the generalization of the Lyapunov exponent calculation to higher dimensions for discrete mappings (Lyapunov exponents for continuous flows are a natural extension of the following discussion). Consider the N-dimensional map, $\mathbf{M}$, such that

$$\mathbf{x}_{p+1} = \mathbf{M}(\mathbf{x}_p)$$

As before we assume there is a periodic orbit of period p such that for a given starting point, $\mathbf{x}_0$, we have

$$\mathbf{x}_p = \mathbf{x}_0 \; with \; \mathbf{x}_p = \mathbf{M}^p(\mathbf{x}_0)$$

To investigate the stability of the orbit we consider a small deviation away from the orbit in the direction of the tangent vector $\mathbf{y}_0$. The tangent vector will evolve after each iteration according to the equation

$$\mathbf{y}_{p+1} = \mathbf{DM}(\mathbf{x}_p) \cdot \mathbf{y}_p$$

where $\mathbf{DM}$ is the Jacobian Matrix for the system and is defined by

$$\mathbf{DM}(\mathbf{x}) = \begin{bmatrix} \frac{\partial M^{(1)}}{\partial x^{(1)}} & \frac{\partial M^{(1)}}{\partial x^{(2)}} & \cdots & \frac{\partial M^{(1)}}{\partial x^{(N)}} \\ \frac{\partial M^{(2)}}{\partial x^{(1)}} & \frac{\partial M^{(2)}}{\partial x^{(2)}} & \cdots & \frac{\partial M^{(2)}}{\partial x^{(N)}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial M^{(N)}}{\partial x^{(1)}} & \frac{\partial M^{(N)}}{\partial x^{(2)}} & \cdots & \frac{\partial M^{(N)}}{\partial x^{(N)}} \end{bmatrix}$$

Looking at the mapping of the initial tangent vector $\mathbf{y}_0$ we have

$$\mathbf{y}_p = \mathbf{DM}^p(\mathbf{x}_0) \cdot \mathbf{y}_0$$

with

$$\mathbf{DM}^p(\mathbf{x}_0) = \mathbf{DM}^{p-1}(\mathbf{x}_{p-1}) \cdot \mathbf{DM}^{p-2}(\mathbf{x}_{p-2}) \ldots \mathbf{DM}(\mathbf{x}_0)$$

We then note that the unit vector for the initial displacement is given by $\mathbf{u}_0 = \frac{\mathbf{y}_0}{|\mathbf{y}_0|}$ and we use this to define the Lyapunov exponent as

$$h(\mathbf{x}_0, \mathbf{u}_0) = \lim_{p \to \infty} \frac{1}{p} ln \left(\frac{|\mathbf{y}_p|}{|\mathbf{y}_0|} \right)$$

$$\Rightarrow h(\mathbf{x}_0, \mathbf{u}_0) = \lim_{p \to \infty} \frac{1}{p} ln \left| \mathbf{DM}^p(\mathbf{x}_0) \cdot \mathbf{u}_0 \right|$$

Note here the key difference between 1D and higher dimensions regarding the dependence of the Lyapunov exponent on the direction via the dependence on $\mathbf{u}_0$.

One might think at this point that Lyapunov exponents do not provide any useful generalization for a chaotic system since the Lyapunov exponent can obtain a different value for each phase space point on the attractor as well as for different directions at the given point. However there is a manipulation that we can peform followed by the use of Oseledecs multiplicative ergodic theorem which will allow us to define a single Lyapunov exponent for *nearly* all phase space points in the basin of attraction of a strange attractor.

To start the manipulation, we take the limit that p is large and define the matrix, $\mathbf{H}_p(\mathbf{x}_0)$ to be $\mathbf{H}_p(\mathbf{x}_0) = [\mathbf{DM}^p(\mathbf{x}_0)]^\dagger[\mathbf{DM}^p(\mathbf{x}_0)]$ then the equation for the Lyapunov exponent becomes

$$h(\mathbf{x}_0, \mathbf{u}_0) \simeq \bar{h}_p(\mathbf{x}_0, \mathbf{u}_0) = \frac{1}{p} ln \left| \mathbf{DM}^p(\mathbf{x}_0) \cdot \mathbf{u}_0 \right|$$

$$\Rightarrow \bar{h}(\mathbf{x}_0, \mathbf{u}_0) = \frac{1}{2p} ln \left| \mathbf{u}_0^\dagger \cdot \mathbf{H}_p(\mathbf{x}_0) \cdot \mathbf{u}_0 \right|$$

where $\bar{h}_p(\mathbf{x}_0, \mathbf{u}_0)$ is an approximation to the actual Lyapunov exponent. Now the matrix $\mathbf{H}_p(\mathbf{x}_0)$ is a real non-negative, Hermitian matrix and therefore it has real non-negative eigenvalues as well as real eigenvectors. Let us then choose $\mathbf{u}_0$ such that is lies in the directions of one of the eigenvectors of $\mathbf{H}_p$. There will N of these eigenvectors which will produce N Lyapunov exponents given by

$$\bar{h}_{jp} = \frac{1}{2n} ln(H_{jp})$$

In the limit as $p \to \infty$ these approximations to the Lyapunov exponents, $\bar{h}_{jp}$, converge to the actual Lyapunov exponents h_j. We then order the Lyapunov exponents from largest (most positive) to smallest (zero or most negative)

$$h_1(\mathbf{x}_0) \geq h_2(\mathbf{x}_0) \geq \cdots \geq h_N(\mathbf{x}_0)$$

With this ordering we proceed by changing our assumption of $\mathbf{u}_0$. Previously we had chosen $\mathbf{u}_0$ to be coincident with one of the eigenvectors of $\mathbf{H}_p$. Now we place no restriction on $\mathbf{u}_0$ and we assume that it is in an arbitrary direction. This means that $\mathbf{u}_0$ can be expressed as a linear combination of the eigenvectors, $\mathbf{e}_j$, of $\mathbf{H}_p$ since the eigenvectors span the space and are assumed to be normalized,

$$\mathbf{u}_0 = \sum_{j=1}^{N} a_j \mathbf{e}_j$$

$$\Rightarrow \mathbf{u}_0^\dagger \cdot \mathbf{H}_p(\mathbf{x}_0) \cdot \mathbf{u}_0 = \sum_{j=1}^{N} a_j^2 \; exp[2p\bar{h}_{jp}(\mathbf{x}_0)]$$

For very large p the sum will be dominated by the largest Lyapunov exponent due to the exponential dependence. Since we ordered the Lyapunov exponents in decreasing order, the dominate Lyapunov exponent is given by $h_1(\mathbf{x}_0)$. Thus for an arbitrary choice of $\mathbf{u}_0$ and for sufficiently large p we have the surprising result that

$$h(\mathbf{x}, \mathbf{u}_0) = h_1(\mathbf{x}_0)$$

Now we use Oseledec's multiplicative ergodic theorem which states that the if there is an ergodic measure μ of a strange attractor then the Lyapunov exponents obtained with respect to that measure are the same for all $\mathbf{x}_0$ up to a set of measure 0 (hence the previous emphasis that *nearly* all the points in the basin of attraction can be assigned a single Lyapunov exponent). Therefore our previous manipulation shows that at a given point, even thought there are possibly different Lyapunov exponents for different choices of direction, to first order for an arbitrary direction, the largest Lyapunov exponent will dominate the dynamics. Then Oseledec's theorem shows that this largest Lyapunov exponent is the same Lyapunov exponent for nearly all $\mathbf{x}_0$ within the basin of attraction of the strange attractor. Hence we can very readily assign a single value of the Lyapunov exponent for a given strange attractor in a chaotic system. Continuing to use the Baker's Map as our canonical example we now calculate the Lyapunov exponent for this map. For reference the Baker's Map is given by

$$x_{p+1} = \begin{cases} \lambda_a x_p, & y_p < \alpha \\ (1-\lambda_a) + \lambda_b x_n, & y_p > \alpha \end{cases}$$

$$y_{p+1} = \begin{cases} y_p/\alpha, & y_p < \alpha \\ (y_p - \alpha)/\beta, & y_p > \alpha \end{cases}$$

with $\beta = 1 - \alpha$ and $\lambda_a + \lambda_b \leq 1$. To calculate the Lyapunov exponent we calculate the terms for the matrix $\mathbf{DM}$:

$$\frac{\partial x_{p+1}}{\partial x_p} = \begin{cases} \lambda_\alpha, & y_p < \alpha \\ \lambda_\beta, & y_p > \alpha \end{cases}$$

$$\frac{\partial x_{p+1}}{\partial y_p} = 0 \quad \frac{\partial y_{p+1}}{\partial x_p} = 0$$

$$\frac{y_{p+1}}{y_p} = \begin{cases} 1/\alpha, & y_p < \alpha \\ 1/\beta, & y_p > \alpha \end{cases}$$

Hence the matrix $\mathbf{H}_p = \mathbf{DM}^\dagger \cdot \mathbf{DM}$, is given by

$$\mathbf{H}(\mathbf{x}) = \begin{bmatrix} (\lambda_a^{p_1}\lambda_b^{p_2})^2 & 0 \\ 0 & (\lambda_a^{-p_1}\lambda_b^{-p_2})^2 \end{bmatrix}$$

where p_1 is the number of times the orbit falls below the horizontal line $y = \alpha$ and p_2 is the number of times the orbit falls above the line. The total number of iterations is then given by $p = p_1 + p_2$. From the structure of the matrix we clearly see that the eigenvectors are along the x-dimension and the y-dimension. Evaluating the Lyapunov exponent along these two dimensions then gives us

$$h(\mathbf{x}_0, \hat{x}) = \lim_{p\to\infty}\left(\frac{p_1}{p}ln(\lambda_a) + \frac{p_2}{p}ln(\lambda_b)\right)$$

$$h(\mathbf{x}_0, \hat{y}) = \lim_{p\to\infty}\left(\frac{p_1}{p}ln\left(\frac{1}{\alpha}\right) + \frac{p_2}{p}ln\left(\frac{1}{\beta}\right)\right)$$

This can be further simplified by noting that the quantity $\lim_{p\to\infty} p_1/p$ is the natural measure for $y < \alpha$ and similarly the quantity $\lim_{p\to\infty} p_2/p$ is the natural measure for $y > \alpha$ therefore

$$\lim_{p\to\infty}\frac{p_1}{p} = \alpha$$

$$\lim_{p\to\infty}\frac{p_2}{p} = \beta$$

Thus the final forms of the Lyapunov exponents for the Baker's Map are given by

$$h_x = \alpha ln(\lambda_a) + \beta ln(\lambda_b)$$

$$h_y = \alpha ln\left(\frac{1}{\alpha}\right) + \beta ln\left(\frac{1}{\beta}\right)$$

This completes our general discussion of Lyapunov exponents. However, as mentioned previously there is a connection between the geometric information dimension and the Lyapunov exponents. This takes the form of the Kaplan-York Conjecture which gives the fractal dimension of a strange attractor in terms of the Lyapunov exponents. Recalling our ordering of the Lyapunov exponents from largest to smallest, we then let K be the largest value such that

$$\sum_{j=1}^{K} h_j \geq 0$$

In order words, this sum is summing up the maximum number of Lyapunov exponents such that the sum still remains greater than or equal to zero (note that this does not exclude negative Lyapunov exponents from the sum). We then define the Lyapunov Dimension, D_L, to be

$$D_L = K + \frac{1}{|h_{K+1}|}\sum_{j=1}^{K} h_j$$

The Kaplan-York conjecture claims that the Lyapunov Dimension is equal to the Information dimension for attractors that are loosely defined as "typical attractors". This result is fairly powerful since it allows for the calculation of a very important dimension of the attractor (information dimension) via a calculation of the more tractable Lyapunov exponents. To demonstrate the Kaplan-York conjecture we once again turn to the Baker's Map. If we assume that $\lambda_a = \lambda_b$ in the Baker's map, then the Lyapunov dimension is given by

$$D_L = 1 + \frac{h_y}{h_x}$$

$$D_L = 1 + \frac{\alpha ln(1/\alpha) + \beta ln(1/\beta)}{(\alpha+\beta) ln(\lambda)}$$

but since $\alpha + \beta = 1$ we have that

$$D_L = 1 + \frac{\alpha ln(\alpha) + \beta ln(\beta)}{|ln(\lambda)|} = D_1$$

which is precisely the information dimension for the Baker's map that was derived in a previous lecture.

Metric Entropy

We now turn our attention to the discussion of entropy as another tool for quantifying chaotic systems. The entropy used here for dynamical systems is called the metric entropy or the Kolmogorov-Sinai entropy, after it originators. Our goal will be to derive the standard form of the metric entropy and then show its relation to the Lyapunov exponents. Our formulation for the metric entropy is based on the Shannon formulation for entropy where H_S is the uncertainty that an event will result if there are r different possibilities with probability $(p_1, p_2, \ldots p_r)$. Then the Shannon entropy is defined as

$$H_S = \sum_{i=1}^{r} p_i ln\left(\frac{1}{p_i}\right)$$

In a similar way for dynamical systems we will derive the metric entropy denoted, $h(\mu)$, using the invariant probability measure μ of the dynamical system. Similar to statistical mechanics, we start by defining a bounded region, W, of phase space and then we define a partition such that W is divided into r disjoint parts.

$$W = W_1 \cup W_2 \cup \ldots W_r$$

Then we use this partition to define the entropy function using the Shannon entropy form

$$H(W_i) = \sum_{i=1}^{r} \mu(W_i) ln\Big(\mu(W_i)^{-1}\Big)$$

Our objective is to construct a series of partitions $\{W_i^{(n)}\}$ that are finer and finer (set size gets smaller and smaller). In order to do this we start with the original partition and take the inverse map of that set $\mathbf{M}^{-1}(W_k)$. Then we create the n^2 intersections $W_j \cap \mathbf{M}^{-1}(W_k)$ by iterating through all the j-w pairs with $j = 1, 2, \ldots r$ and $k = 1, 2, \ldots r$. The intersections of these two sets forms out next partition, $\{W_i^{(2)}\}$. Notice that in doing this process we have taken a set with r elements and intersected it with another set of r elements to form a set that has r^2 elements and is therefore a finer partition than either of our starting partitions. This process can be repeated n times until we have the set with r^n elements

$$\{Wi^{(n)}\} = W_{i_1} \cap \mathbf{M}^{-1}(W_{i_2}) \cap \mathbf{M}^{-2}(W_{i_3}) \cap \cdots \cap \mathbf{M}^{-(n-1)}(W_{i_n})$$

Using this nth iteration we can then write the entropy equation using the Shannon form as

$$h(\mu, \{W_i\}) = \lim_{n\to\infty} \frac{1}{n} H(\{W_i^{(n)}\})$$

The metric entropy itself is obtained when we maximize the equation over all the possible initial partitions. Therefore the metric entropy, $h(\mu)$, is given by

$$h(\mu) = \sup_{W_i} h(\mu, W_i)$$

To show a calculation of the metric entropy we once again returning to our discussion of the Baker's Map. For the Baker's map, we use the following partition as the first partition of the phase space

$$H(\{W_i\}) = \alpha ln(1/\alpha) + \beta ln(1/\beta)$$

It can then be shown that successive processes of creating finer and finer partitions results in $\{W_i^{(n)}\}$ such that

$$H(\{W_i^{(n)}\}) = n\big(\alpha ln(1/\alpha) + \beta ln(1/\beta)\big) = nH(\{W_i\})$$

Substituting this result into the equation for the metric entropy gives us

$$h(\mu, \{W_i\}) = \lim_{n\to\infty} \frac{1}{n} n\big(\alpha ln(1/\alpha) + \beta ln(1/\beta)\big)$$

$$h(\mu, \{W_i\}) = \alpha ln(1/\alpha) + \beta ln(1/\beta)$$

As noted previously, in order to obtain the actual metric entropy we must maximize the entropy equation over all possible $\{W_i\}$. However it can be shown that our original choice for the initial partition above is in fact the partition that maximum the entropy equation, therefore the metric entropy for the Baker's Map is given by

$$h(\mu) = \alpha ln(1/\alpha) + \beta ln(1/\beta)$$

Upon deriving this metric entropy we notice immediately that it is identical to the Lyapunov exponent for the y-dimension which was derived in the previous section:

$$h(\mu) = h_y = \alpha ln(1/\alpha) + \beta ln(1/\beta)$$

Once might then ask if this is a coincidence or indicative of a more general connection between Lyapunov exponents and metric entropy. It turns out that it has been proven that the metric entropy is at most the sum of the positive Lyapunov exponents of the chaotic system

$$h(\mu) \leq \sum_{h_i>0} h_i$$

For the Bakers map there is only one positive Lyapunov exponent and therefore the equality holds rather than the inequality. In addition it has also been shown that for a Hamiltonian system the metric entropy is exactly equal to the positive Lyapunov exponents.

$$h(\mu) = \sum_{h_i>} h_i$$

Thus in conclusion we have seen that there is a strong connected between the dimension, Lyapunov and entropy descriptions of a chaotic system and, generally speaking, knowledge of one of these descriptions allows us to determine some information about the other two.

References

[1] Ott, Edward. "Chaos in Dynamical Systems." 2nd Edition. Cambridge University Press, 2002.