Phase Turbulence

Have been concerned with phase-diffusion equation:

$$\frac{\partial \phi}{\partial t} = \omega(x) + \alpha \nabla^2 \phi + \beta (\nabla\phi)^2$$

Have considered: - derivation
- 1D solutions
- spiral waves (phase singularities)

Now: turbulence? ....

Aside: - What does "turbulence" mean? .....
- Operationally, here "turbulence" means at least 1 positive Lyapunov exponent (N.B. Phase dynamics $\Rightarrow$ all $\lambda \leq 0$ i.e. $\lambda_\phi = 0$, $\lambda_\perp < 0$).

- Routes: - instability and turbulent saturation
- defect interaction/mediation (i.e. spirals, vortices).

Ref.: "Dynamical Systems Approach to Turbulence"
T. Bohr, M.H. Jensen, G. Paladin, A. Vulpiani (Cambridge U. Press)

For stability, return to CGL:

$$\partial_t A = A - (1+i\alpha)|A|^2 A + (1+i\beta)\nabla^2 A$$

NL freq. shift dispersion

(i.e. can normalize time to growth rate)

$$A = Re^{i\phi} \Rightarrow$$

$$(iR\,\partial_t\phi + \partial_t R) = R - R^3 - i\alpha R^3 + (1+i\beta)\big(iR\nabla^2\phi - R(\underline{\nabla}\phi)^2 + 2i\,\underline{\nabla}\phi\cdot\underline{\nabla}R + \nabla^2 R\big)$$

$\Rightarrow$ real and imaginary parts:

$$\partial_t R = R - R^3 - \beta R\nabla^2\phi - 2\beta\,\underline{\nabla}R\cdot\underline{\nabla}\phi - R(\underline{\nabla}\phi)^2 + \nabla^2 R$$

$$R\,\partial_t\phi = -\alpha R^3 + R\nabla^2\phi + 2\,\underline{\nabla}R\cdot\underline{\nabla}\phi - \beta R(\underline{\nabla}\phi)^2 + \beta\nabla^2 R$$

Now, for linear stability:

$$A = (1+\rho(\underline{x},t))\,e^{i(\phi(\underline{x},t)-\alpha t)}$$

i.e. $R = 1 + \rho(\underline{x},t)$

$\Phi = (\phi(\underline{x},t) - \alpha t)$

linearizing:

$$\partial_t (1+\rho) = (1+\rho) - (1+\rho)^3 - \beta(1+\rho)\nabla^2\phi$$
$$-2\beta\,\nabla\rho\cdot\nabla\phi + \nabla^2(1+\rho)$$

$$\partial_t \rho \cong -2\rho + \nabla^2\rho - \beta\nabla^2\phi$$

$$\partial_t \phi \cong \nabla^2\phi + \beta\nabla^2\rho - 2\alpha\rho$$

$$\therefore \boxed{\begin{aligned}\partial_t \rho &\cong -2\rho + \nabla^2\rho - \beta\nabla^2\phi \\ \partial_t \phi &\cong \nabla^2\phi + \beta\nabla^2\rho - 2\alpha\rho\end{aligned}}$$

$$\begin{pmatrix}\rho\\ \phi\end{pmatrix} = \begin{pmatrix}\rho_0\\ \phi_0\end{pmatrix} e^{i(\underline{k}\cdot\underline{x} - \omega t)}$$

$$-i\omega\,\rho_0 \cong -2\rho_0 - k^2\rho_0 + \beta k^2\phi_0$$

$$-i\omega\,\phi_0 \cong -k^2\phi_0 - k^2\beta\,\rho_0 - 2\alpha\rho_0$$

$$\Rightarrow \left(-i\omega + 2 + k^2\right)\left(-i\omega + k^2\right) + \beta k^2\left(2\alpha + \beta k^2\right)$$

$$\therefore \; -i\omega = \gamma$$

$$\gamma = -(1+k^2) \pm (1 - 2\alpha\beta k^2 - \beta^2 k^4)^{1/2}.$$

thus, note for (+) root:

$$\boxed{\gamma \underset{k \to 0}{\cong} -(1+\alpha\beta)k^2 - \frac{\beta^2}{2}(1+\alpha^2)k^4 + \ldots}.$$

(1)

$\therefore \Rightarrow$ long wavelength instability if:

$$1 + \alpha\beta < 0$$

$\leftrightarrow$ negative diffusion instability

i.e. in general;

$$\partial_t \phi = a\nabla^2\phi + b(\underline{\nabla}\phi)^2 + \ldots$$

$$a = 1 + \alpha\beta$$

$\alpha \to$ NL frequency shift
$\beta \to$ NL dispersion

Homework # 6: Show this, via derivation of phase equation.

② $\gamma \cong -(1+\alpha\beta)k^2$

$$(\gamma + 2 + k^2)\,\rho_0 \cong \beta k^2 \phi_0$$

$$\therefore \quad \rho_0 \cong \frac{\beta k^2}{\gamma + 2 + k^2}\,\phi_0$$

__but__ phase instability appears at long wavelength

i.e. $k < k_{crit} \approx \left(\dfrac{2\,|1+\alpha\beta|}{\beta^2 (1+\alpha^2)} \right)^{1/2}$

$\Rightarrow |\rho_0| \ll |\phi_0| \Rightarrow$ phase fluctuations substantially larger!

$\therefore$ name "phase turbulence".

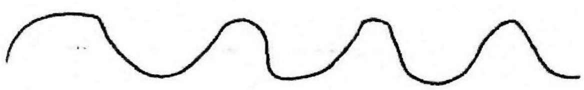
③ Recall notions of "attractive" and "repulsive" coupling, i.e. (pgs. 50 – 55)

i.e. $\dfrac{d\psi}{dt} = -\nu + \epsilon \sin\psi$ ("ψ" is actually $\psi_1 - \psi_2$)

Aside: Modulational Instability

The negative viscosity instability of the phase diffusion equation is an example of the Benjamin-Feir or modulational instability.

Consider a wavetrain:



k, $\omega(k)$ — carrier
amplitude a
$\mathcal{E} \sim a^2$
wave energy density

then can write:

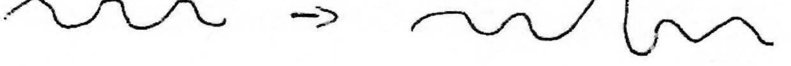
$$\frac{dk}{dt} = -\frac{\partial \omega}{\partial x} \qquad \text{(eikonal theory)}$$

$$\frac{\partial}{\partial t}\mathcal{E} + \frac{\partial}{\partial x}\left(v_{gr}\,\mathcal{E}\right) = 0 \qquad \text{(Poynting flux relation)}$$

now, if finite amplitude frequency shift:

$$\omega = \omega_0(k) + \omega_2(k)\, a^2$$

Investigate modulations of train, i.e.:



grow / decay ?

then, can write:

$$\frac{\partial k}{\partial t} + V_g \frac{\partial k}{\partial x} = -\frac{\partial}{\partial x}\left(\cancel{\omega_0(k)} + \omega_2(k)\, a^2\right)$$

$$\frac{\partial}{\partial t} a^2 + V_g \frac{\partial}{\partial x} a^2 + \frac{\partial^2 \omega_0}{\partial k^2} a^2 \frac{\partial k}{\partial x} = 0$$

$$\left.\begin{aligned} \delta a^2 &= a_0^2\, e^{i(qx - \Omega t)} \\ \delta k &= k_0\, e^{i(qx - \Omega t)} \end{aligned}\right\} \text{modulational perturbations} \quad q \ll k, \ \Omega \ll \omega$$

$\Rightarrow$ linearizing:

$$\frac{\partial}{\partial t}\delta k + V_g \frac{\partial}{\partial x}\delta k + \omega_2(k) \frac{\partial}{\partial x}\delta a^2 = 0$$

$$\frac{\partial}{\partial t}\delta a^2 + V_g \frac{\partial}{\partial x}\delta a^2 + V_g' a_0^2 \frac{\partial \delta k}{\partial x} = 0$$

$$-i(\Omega - q V_g)\,\delta k + i q\, \omega_2(k)\, \delta a^2 = 0$$

$$i q a_0^2 V_g' \delta k + -i(\Omega - q V_g)\,\delta a^2 = 0$$

$$\therefore \; -(\Omega - q V_g)^2 + q^2 V_g' \omega_2(k) a_0^2 = 0$$

$$\Omega - q V_g = \pm \left(q^2 \frac{\partial^2 \omega}{\partial k^2} \omega_2(k) a_0^2 \right)^{1/2}$$

$\Rightarrow$ - modulational perturbation grows for

$$\omega_2(k) \frac{\partial^2 \omega}{\partial k^2} < 0$$

- growth $\sim a_0$ (amplitude dependent)

- can be stabilized by diffraction (beyond eikonal theory)

- can saturate by ray trapping, wave breaking etc.

- generic mechanism (model independent) $\Rightarrow$ first discovered for water wave trains in '67 by Benjamin and Feir.

- can note parallel:

$V_g' \omega_2 < 0 \iff \alpha\beta < 0 \quad (1 + \alpha\beta < 0)$

$\omega_2 \to$ NL ω shift $\iff \beta$ in $\beta (\nabla\phi)^2$

$V_g' \to \omega'' \iff \alpha \to Z''$

so $\epsilon < 0$: $\psi = 0$ is stable as $r \to 0$
$\Rightarrow$ "attractive" coupling

$\epsilon > 0$: $\psi = \pi$ is stable as $r \to 0$
$\Rightarrow$ "repulsive" coupling — oscillators get out of phase.

and recall for coupled L-S. oscillators:

$$\dot{A}_{\substack{1\\2}} = -i\Delta_{\substack{1\\2}} A_{\substack{1\\2}} + \mu_{\substack{1\\2}} A_{\substack{1\\2}} - (\gamma_{\substack{1\\2}} + i\alpha_{\substack{1\\2}})|A_{\substack{1\\2}}|^2 A_{\substack{1\\2}}$$
$$+ \underbrace{(\beta + i\delta)}_{\text{dissipative coupling}}{}_{\substack{1\\2}} (A_{\substack{2\\1}} - A_{\substack{1\\2}})$$

reactive coupling

and recall:

$$\dot{\psi} = -\nu - 2(\beta + \alpha\delta)\sin\psi$$

<u>So</u> $\beta + \alpha\delta > 0$ $\to$ attraction

$\beta + \alpha\delta < 0$ $\to$ repulsion

Now, translating the notation:

$\beta = 1$ for CGL used here
(does not correspond to β)

$\alpha \Longleftrightarrow \alpha$, i.e. in both cases, α corresponds to NL frequency shift

$\delta \Longleftrightarrow \beta$, i.e. δ and α both refer to reactive oscillator-to-oscillator coupling.

thus,

$$\beta + \alpha\delta \rightarrow 1 + \alpha\beta$$

so → positive phase diffusion: $1 + \alpha\beta > 0$

⇒ attractive coupling.

→ negative phase diffusion: $1 + \alpha\beta < 0$

⇒ repulsive coupling.

→ For $1+\alpha\beta < 0 \Rightarrow$ phase instability

$\therefore$ natural to examine effect of retaining "lowest/dominant" nonlinear terms.

$\Rightarrow$ Kuramoto – Sivashinsky Equation !

i.e. can write answer, from experience with phase diffusion equation, i.e. :

$$\frac{\partial}{\partial t}\phi = \underbrace{-D\nabla^2\phi}_{\substack{D = 1+\alpha\beta < 0 \\ \text{(negative diffn)}}} + \underbrace{\lambda \underline{\nabla}\phi \cdot \underline{\nabla}\phi}_{\substack{\text{KPZ-Burgers} \\ \text{nonlinearity}}} \underbrace{-\gamma \nabla^2\nabla^2\phi}_{\substack{\text{high } k \\ \text{cut-off} \\ \text{(ad-hoc)}}}$$

N.B. Obvious K-S. equation is variant member of KPZ – Burgers family.

Now to derive Kuramoto – Sivashinsky Equation :

$\Rightarrow$ recall equations for R, Φ.

Now, convenient to re-write for:

$$\rho = 1 - R$$

$$\phi = \Phi + \alpha t$$

$\Rightarrow$

$$\partial_t \rho = -2\rho - 3\rho^2 - \rho^3 + \nabla^2\rho - \beta(1+\rho)\nabla^2\phi - 2\beta\, \underline{\nabla}\rho \cdot \underline{\nabla}\phi - (1+\rho)(\underline{\nabla}\phi)^2$$

$$-(1+\rho)\partial_t\phi = -2\alpha\rho - 3\alpha\rho^2 - \alpha\rho^3 + \beta\nabla^2\rho + (1+\rho)\nabla^2\phi + 2\,\underline{\nabla}\rho\cdot\underline{\nabla}\phi - \beta(1+\rho)(\underline{\nabla}\phi)^2$$

and also recall:

- $|\hat{\rho}|/|\hat{\phi}| \sim |\beta k^2|$, with long wavelength instability favored. (phase turbulence)

$\therefore$ $k \sim O(\epsilon)$

$\omega \sim O(\epsilon^2)$

- $|\hat{\rho}|/|\hat{\phi}| \sim O(\epsilon^2)$

so keeping terms to $O(\epsilon) \Rightarrow$

$$0 = -2\rho - \beta \nabla^2\phi - (\underline{\nabla}\phi)^2$$

$$\partial_t \phi = -2\alpha\rho + \nabla^2\phi - \beta(\underline{\nabla}\phi)^2$$

$\Rightarrow$

$$= -\frac{2}{2}\alpha\left(-\beta\,\nabla^2\phi - (\underline{\nabla}\phi)^2\right) + \nabla^2\phi - \beta(\underline{\nabla}\phi)^2$$

$$\therefore \quad \boxed{\partial_t \phi = (1+\alpha\beta)\nabla^2\phi + (\alpha-\beta)(\underline{\nabla}\phi)^2}$$

phase diffusion equation.

if $1+\alpha\beta < 0$, need retain higher-k terms, i.e.

$$\partial_t \phi = -|(1+\alpha\beta)|\,\nabla^2\phi + (\alpha-\beta)(\underline{\nabla}\phi)^2 - \mu\nabla^4\phi$$

$$\mu = (\beta^2/2)(1+\alpha^2), \qquad \lambda = \alpha - \beta$$

$$\boxed{\partial_t \phi = -D\,\nabla^2\phi + \lambda(\underline{\nabla}\phi)^2 - \mu\,\nabla^4\phi}$$

K-S equation

Note: K-S equation is typical of KPZ / Burgers family

i.e.

KPZ:

$$\partial_t \phi - D \nabla^2 \phi = \lambda (\nabla \phi)^2 + \tilde{f}_N \quad \left(\text{forced by noise}\right)$$

Burgers: $\underline{V} = \nabla \phi$

$$\partial_t \underline{V} + \underline{V} \cdot \nabla \underline{V} = D \nabla^2 \underline{V} + \tilde{\underline{f}}_N \left(\text{I UP or noise forcing}\right)$$

Kuramoto - Sivashinsky:

$$\partial_t \phi = -D \nabla^2 \phi - \mu \nabla^4 \phi + \lambda (\nabla \phi)^2$$

- negative viscosity, positive hyper-viscosity
- nonlinearity generates/couples to smaller scales $\Rightarrow$ hyperviscosity can maintain stationarity as small scale cut-off.