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Calculation of the Kolmogorov Entropy for Motion Along a Stochastic Magnetic Field

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An expression for the Kolmogorov entropy has been derived. Excellent agreement between a probability description and direct dynamical computations has been found.

A typical magnetic field of interest to magnetic confinement physics may be taken to be of the form

$$\vec{B} = B_0\hat{z} + B_y(x)\hat{y} + \delta\vec{B}, \tag{1}$$

where $|\delta\vec{B}| \ll B_y, B_0$ might result from any short-wavelength plasma microinstabilities. When δB exceeds a very small value such that neighboring magnetic islands overlap, the field structure becomes stochastic, and can be characterized by two geometrical properties.[1] First, a given field line diffuses in x, and, second, two neighboring field lines diverge from each other—the mean distance between them increasing as $d \sim \exp(z/L_c)$. The diffusion of a field line has obvious implications for the confinement of particles orbiting along the field and has been calculated using quasilinear theory.[2] However, the divergence of field lines is also important in calculating transport since it prevents the reversible wandering of a particle back and forth along a single field line as velocity is reversed by collisions.[1] The value $1/L_c$ is called the Kolmogorov entropy and is defined by the formula[3]

$$h = \lim_{z\to 0}\lim_{d_0\to 0}\left[\frac{1}{z}\ln\left(\frac{d(z)}{d_0}\right)\right]. \tag{2}$$

Since exponential divergence is a statistical property, one expects that averaging along trajectories ($z\to\infty$) can be replaced by phase-space averaging with a proper distribution function. It is our purpose in this note to introduce such a statistical description and to give a derivation of the dependence of h on the properties of the fields given by Eq. (1). Our main result, Eq. (19), is similar to that already obtained by Krommes, Kleva, and Oberman,[4] but we have tried to make the basic assumptions more transparent and the derivation more quantitative. We have also developed a simplified model of a stochastic field for which we have computed h directly using Eq. (2). This model allows detailed numerical study which agrees very well with the theoretical predictions.

Consider a magnetic field given by Eq. (1). In order to model poloidal and toroidal periodicity we assume that the system is periodic in the y direction with period $2\pi a$ and in the z direction with period $2\pi R$. Then $\delta\vec{B}$ can be written in the form

$$\delta\vec{B}=B_0\sum_{m,n}\vec{b}_{mn}(x)\exp[i(my/a-nz/R)]+\text{c.c.} \quad (3)$$

The turbulent nature of $\vec{b}_{mn}$ is introduced by assuming that they have random phases but constant, saturated, amplitudes. Equations for the coordinates of the field lines are

$$dx/dz=B_x/B_0;\quad dy/dz=B_y/B_0. \quad (4)$$

Assuming shear, $L_s^{-1}=B_0^{-1}dB_y/dx$, being constant, we can write (4) in the dimensionless form

$$dx'/dz'=b;\quad dy'/dz'=x'. \quad (5)$$

Here $b=\delta B_x/B_0$ is a small parameter in our problem and $x'=x/L_s$, $y'=y/L_s$, $z'=z/L_s$. We will omit primes in the following. In order to calculate h we have to find the divergence rate of two arbitrarily close particle trajectories. Equations for the differences in position $X=x_2-x_1$ and $Y=y_2-y_1$ can be easily written using Eq. (5):

$$dx/dz=(\partial b/\partial x)X+(\partial b/\partial y)Y,\quad dY/dz=X. \quad (6)$$

It is convenient to put $X=\Lambda Y$ so that

$$d\Lambda/dz=-\Lambda^2+(\partial b/\partial x)\Lambda+\partial b/\partial y;$$
$$dY/dz=\Lambda Y. \quad (7)$$

Since it will turn out later that the characteristic width of all functions with respect to Λ is $\Lambda\sim h\sim b^{2/3}$, we may neglect the term $\Lambda\partial b/\partial x$ if $h\ll 1$. The same condition allows us to neglect the contribution from the term δB_y in comparison with the shear term in Eq. (6). The case of zero shear has been recently considered by Kadomtsev and Pogutse.[5] Thus the basic limitation of our theory is determined by the condition $h\ll 1$. Note that we use here dimensionless h appropriate to Eq. (5).

Consider a number of test particles which are distributed initially with the probability $P(x_i,0)$ (two-point distribution function which is assumed to be a smooth function) in a cross section $z=0$, x_i is a vector with the components (x,y,Λ,Y). If these particles are moving freely along the field lines remaining in the same cross section z, then the evolution of their distribution function $P(x_i,z)$ is described by the continuity equation

$$\partial P/\partial z+\sum_{i=1}^{4}\partial(v_iP)/\partial x_i=0, \quad (8)$$

where v_i is a generalized velocity with components $(b,x,-\Lambda^2+\partial b/\partial y,\Lambda Y)$ and z coordinate plays the role of time. With the use of a statistical description, the Kolmogorov entropy can be defined by the formula

$$h=(\partial/\partial z)[\tfrac{1}{2}\int\ln(X^2+Y^2)\langle P\rangle\,d\Lambda\,dY], \quad (9)$$

where $\langle P\rangle$, which is a function only of Λ, Y, and z, is a probability averaged over x and y which varies slowly in z. It satisfies the equation

$$\frac{\partial\langle P\rangle}{\partial z}-\frac{\partial}{\partial\Lambda}(\Lambda^2P)+\left\langle\frac{\partial b}{\partial y}\frac{\partial P}{\partial\Lambda}\right\rangle+\frac{\partial}{\partial Y}(\Lambda Y\langle P\rangle)=0. \quad (10)$$

We derive this equation by averaging Eq. (8). The fluctuating part $\tilde{P}=P-\langle P\rangle$ is small and satisfies a first-order equation

$$\frac{\partial\tilde{P}}{\partial z}+\frac{\partial}{\partial y}(x\tilde{P})-\frac{\partial}{\partial\Lambda}(\Lambda^2\tilde{P})+\frac{\partial}{\partial y}(\Lambda Y\tilde{P})=-\frac{\partial b}{\partial x}\langle P\rangle-\frac{\partial b}{\partial y}\frac{\partial\langle P\rangle}{\partial\Lambda}. \quad (11)$$

We may neglect the third and fourth terms in this equation because $\Lambda\ll 1$. We can now solve Eq. (11) taking $\partial\langle P\rangle/\partial\Lambda$ to be slowly varying because of the smallness of b and substitute $\tilde{P}$ into (10) to get

$$\frac{\partial\langle P\rangle}{\partial z}-\frac{\partial}{\partial\Lambda}(\Lambda^2\langle P\rangle)+\frac{\partial}{\partial Y}(\Lambda Y\langle P\rangle)-D_{ef}\frac{\partial^2\langle P\rangle}{\partial\Lambda^2}=0, \quad (12)$$

$$D_{ef}=\pi\frac{RL_s}{a^2}\sum_{m,n}\langle|mb_{mnx}(x)|^2\delta(mxR/a-n)\rangle. \quad (13)$$

These calculations are very similar to the derivation of the magnetic diffusion coefficient in quasilinear theory.[2] Notice that the first term on the right-hand side of Eq. (11) averages to zero. We will deal later only with $\langle P\rangle$ and suppress the angular brackets hereafter.

A general initial-value solution for $P(\Lambda,Y,z)$ is impractical to obtain. It is sufficient to look for moments. Let us define

$$P_0(\Lambda,z)=\int_{-\infty}^{\infty}P\,dY$$

and

$$P_e(\Lambda,z)=\tfrac{1}{2}\int_{-\infty}^{\infty}(\ln Y^2)P\,dY;$$

then

$$\frac{\partial P_0}{\partial z}-\frac{\partial}{\partial\Lambda}(\Lambda^2P_0)-D_{ef}\frac{\partial^2P_0}{\partial\Lambda^2}=0 \quad (14)$$

and

$$\frac{\partial P_e}{\partial z} - \frac{\partial}{\partial \Lambda}(\Lambda^2 P_e) - D_{ef}\frac{\partial^2 P_e}{\partial \Lambda^2} = \Lambda P_0. \quad (15)$$

The steady-state solution of Eq. (14) valid for large z, and well behaved at $\Lambda = \pm\infty$, is

$$P_0(\Lambda) = C\exp(-\Lambda^3/3D_{ef}) \times \int_{-\infty}^{\Lambda}\exp(\Lambda'^3/3D_{ef})\,d\Lambda', \quad (16)$$

where C is a normalization constant determined from the condition $\int P_0\,d\Lambda = 1$. We may now solve for

$$P_e = hzP_0(\Lambda) + Q(\Lambda), \quad (17)$$

where Q obeys

$$\frac{d}{d\Lambda}(\Lambda^2 Q) + D_{ef}\frac{d^2Q}{d\Lambda^2} = (h - \Lambda)P_0, \quad (18)$$

with the boundary conditions $Q(\pm\infty) \to 0$. We note that the constant h is just the Kolmogorov entropy as defined by Eq. (9). It can be determined by the solvability condition for Eq. (18). This may be obtained by integrating Eq. (18) over Λ from $-\infty$ to $+\infty$. The left-hand side of Eq. (18) is a full derivative and gives zero. Hence we deduce that h is given by

$$h = \frac{1}{4}\frac{(\frac{2}{3})!}{(\frac{1}{3})!}(3D_{ef})^{1/3}. \quad (19)$$

We should mention here that in all previous formulas we have to use a cutoff parameter $\pm\Lambda_0$ $(1 \gg \Lambda_0 \gg h)$ instead of $\pm\infty$. But because all integrals are rapidly converging within region of $|\Lambda| \lesssim h$ we can formally extend the integration to $\pm\infty$. Going back to dimensional units we can write $L_c = L_s/h$. The basic approximation in our theory can also be written as $L_s/L_c \ll 1$.

Let us turn now to the results of our numerical computations. We used the following dimensionless units: $2\pi a = 1$, $2\pi R = 1$, $L_s = 1$, and took $b_{mnx} = \epsilon\exp(i2\pi m\varphi_{mp})/2i$, $m = 1, \ldots, M$, $p = n \bmod N$, $n = 0 \pm 1, \pm 2, \ldots$, where phases φ_{mp} are randomly choosen numbers between 0 and 1. Summing over n with fixed p in Eq. (3) gives us δ functions, allowing differential Eq. (5) to be reduced to a simple mapping:

$$y_{i+1} = y_i + x_i/N, \qquad x_{i+1} = x_i + f(y_{i+1})/N. \quad (20)$$

Here

$$f(y_{i+1}) = \epsilon = \sum_{m=1}^{M}\sum_{p=0}^{n-1}\sin 2\pi[m(y_{i+1} + \varphi_{mp}) - (i+1)p/N]$$

and one step corresponds to $\Delta z = 1/N$. Equations (13) and (19), applied to this specific model, give

$$D_{ef} = \pi^2\epsilon^2\sum_{m=1}^{M} m^2 \approx \pi^2\epsilon^2 M^3/3, \quad M \gg 1,$$

and

$$h_{\text{th}} = 0.54M\epsilon^{2/3}. \quad (21)$$

On the other hand, h has been calculated directly using Eqs. (20) and the numerical method described by Casartelli *et al.*[6] To obtain reasonable accuracy approximately 10^4 steps were necessary. The results of these computations are presented in Fig. 1. One can see a remarkable agreement with the theoretical formula (21). We have found that h is equal to zero below the stochastic transition (where there exist good magnetic surfaces) and has a well-defined positive value above the stochasticity threshold. This value is independent of initial conditions, that is, it appears numerically that h has the same value for almost all trajectories. Stochastic transition takes place at $\epsilon_M \approx 2/M^3$, in good agreement with the resonance overlapping criterion.[1,3] The results do not depend on the step size $\Delta z = 1/N$ in the region $h < N$, but h does depend on N when $h > N$, and scales in a completely different way with ϵ and M in this region[7]: $h \sim \ln(M^3\epsilon^2)$. In order to derive a formula for h in this case, let us consider eigenvalues of tangential transformation for the mapping (20),[3]

$$\lambda^{\pm} = 1 + K/2 \pm (K^2/4 + K)^{1/2}, \quad (22)$$

where $K = f'(y_{i+1})/N^2$. If $|K| \gg 1$ then $\lambda^+ \approx K$ and

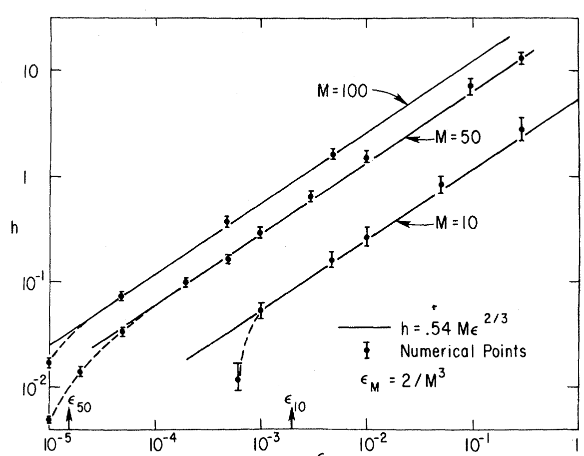

FIG. 1. The numerically obtained Kolmogorov entropy. The solid curve is the theoretical result. Also shown are the stochastic transition points ϵ_M.

divergence takes place preferentially in the x direction. The Kolmogorov entropy can be easily written in this case[3]

$$h = \tfrac{1}{2}N\langle \ln K^2 \rangle_{\text{trajectory}}, \tag{23}$$

The angular brackets here mean averaging along the trajectory. Because $M \gg 1$ and φ_{mp} are random we can make use of a central-limit theorem and perform averaging in Eq. (23) with the Gaussian distribution

$$P(K) = \exp(-K^2/2\langle K^2\rangle_{\text{trajectory}}/(2\pi\langle K^2\rangle_{\text{trajectory}})^{1/2},$$

where $\langle \overline{K}^2\rangle_{\text{trajectory}} = (2/3)\pi^2\epsilon^2(M/N)^3$. After a simple calculation we can get from (23)

$$h = \tfrac{1}{2}N\ln[\tfrac{1}{3}(\pi^2)e^{-c}(M/N)^3\epsilon^2]. \tag{24}$$

Here $c = 0.577$ is the Euler constant. This formula is also in very good agreement with computer calculations.[7] Using Eq. (21), we can rewrite Eq. (24) as $h = \frac{3}{2}N\ln(2.3h_{\text{th}}/N)$, applicable when $h_{\text{th}} > h > N$. But in the region $h_{\text{th}} < N$ we have to use Eq. (21).

We believe that our results have some general interest: They illustrate how a statistical description can be introduced in a deterministic but stochastically unstable system.

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[7]More detailed results of these computations will be published elsewhere.