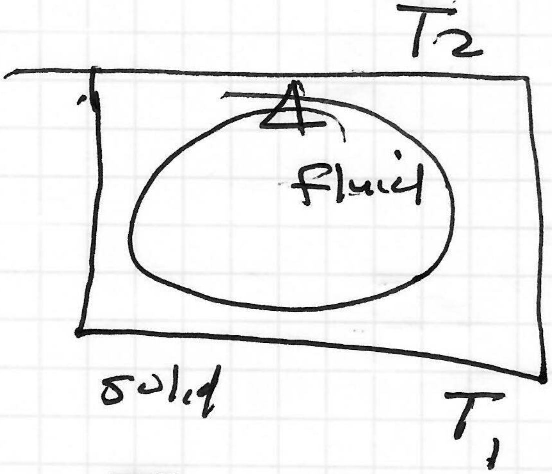
Convection - Fixed Flux *

→ Chapman - Proctor Model (cf Rieutord).

Till now:



$T_1 > T_2$

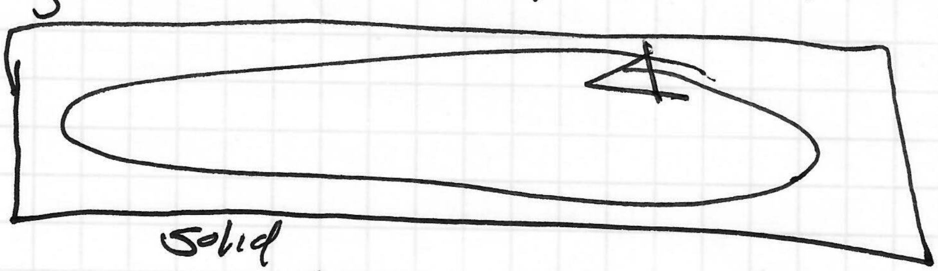
- fixed temperature on bndry.
- assumes box conducts heat faster than fluid ⇒ temp on bndy adjusts to fluctns in fluid.

$\overline{T_s} = T_f$

$$\chi_f \frac{d\overline{T}_f}{dz} = \chi_s \frac{d\overline{T}_s}{dz}$$

(conv.)

Now, consider opposite limit:



Q, $\nabla_z \overline{T}$ fixed.

- here $\chi_s / \chi_f \to 0$ i.e. solid is ideal insulator
- temp in solid fixed ⇒ temp. fluctuations don't propagate into solid

- so, require: $T = T_0 + \Theta$

$$\left.\frac{\partial \Theta}{\partial z}\right|_{0,1} = 0 \quad \Rightarrow$$ no ∇T fluctuations on bndry. (fixed flux).

(bttm, top)

$\Rightarrow \quad Nu = 1$

$$Nu = \frac{Q_{top}}{-\chi_p \, d\bar{T}/dz} = 1$$

- have long, thin slot conducting heat (maintaining fixed flux)

$\Rightarrow$ focus on horizontal pattern dynamics, cell structure

$\Rightarrow$ look for weakly NL solutions.

Chapman–Proctor is another rather lengthy crank. $\Rightarrow$ $\left\{ \begin{array}{l} \text{omit,} \\ \text{due time} \end{array} \right.$

start:

usual

$$\left\{ \begin{aligned} &\frac{\partial}{\partial T}\nabla^2\phi - P_r\, Ra\, \frac{\partial \theta}{\partial X} - P\nabla^2\nabla^2\phi = \{\nabla^2\phi, \phi\} \\ &\frac{\partial \theta}{\partial T} - \frac{\partial \phi}{\partial X} - \nabla^2\theta = J(\theta, \phi) = \{\theta, \phi\} \end{aligned} \right.$$

Order: $\Rightarrow$ long, thin box.

$$\left. \begin{aligned} &\partial_x \rightarrow \epsilon\, \partial_X \\ &\partial_t \rightarrow \epsilon^4\, \partial_T \end{aligned} \right. \Rightarrow \left\{ \begin{array}{l} \text{slow, thin} \\ \text{weakly supercritical} \\ \text{cells.} \end{array} \right.$$

$$\phi \rightarrow \epsilon\, \phi$$

$$Ra = Ra_{crit} + \epsilon^2 \underbrace{\mu^2}_{\text{excedance}}$$

So Reductive P.T. crank $\Rightarrow$

$\Theta_0 = f(X,T)$

$\to$ vertical structure

$\phi_0 = R_{acrit}\, P(z)\, f'(X,T)$

$f(X,T) \to$ horizontal cell structure.

envelope.

$\Rightarrow$ Chapman–Proctor Eqn:

Contrast: Long wavelengths at most unstable

$$\boxed{\partial_T f + A\mathcal{U}^2 \partial_X^2 f + B\,\partial_X^4 f + C\left((\partial_X f)^3\right)' + D\left(f'f''\right)' = 0}$$

$A > 0 \to$ negative diffy

$B > 0 \to$ hyper diffy > 0

can re-write via ∂_X: $\partial_X f \to g$

$$\boxed{\partial_T g + \underbrace{A\mathcal{U}^2 \partial_X^2 g}_{\text{negative diffu.}} + B'\partial_X^4 g + C\left(g^3\right)'' = 0}$$

$\Rightarrow$ Cahn–Hilliard Equation.

(i.e. phase separation) negative diffusion i.e. for CH:

$$(\psi = \rho_2 - \rho_1)$$

$$\partial_t \psi + \underline{v} \cdot \underline{\nabla} \psi = D \nabla^2 \left(-\psi + \psi^3 + \varepsilon^2 \nabla^2 \psi \right)$$

$$\begin{cases} \partial_t \psi + \underline{\nabla} \cdot \underline{J} = 0 \\ \underline{J} = - D \nabla \mu \\ \mu = \left(-\psi + \psi^3 + \varepsilon^2 \nabla^2 \psi \right) \end{cases}$$

$\mu \rightarrow$ chem pot.

$-\psi + \psi^3 \rightarrow \delta F / \delta \psi$

$\varepsilon^2 \nabla^2 \psi \rightarrow$ regularity.

$$F = a(T - T_0) \frac{\psi^2}{2} + \frac{b}{4} \psi^4$$

$$T < T_0$$

→ C-H eqn → coalescence of blobs separated phase.

→ C-P:

counter-rotating cell pair is final, ~ stationary state

Rieutord on C.-P. 6.

7.8 Fixed Flux Convection ♠

7.8.1 Introduction

In all the foregoing matter, we considered that the plates limiting the fluid are perfect heat conductors, so that their temperature remained constant (fixed by a thermostat). Hence, we demanded that the temperature fluctuations vanished on the boundaries.

In the case of a laboratory experiment the boundary conditions are not so simple. As mentioned in Sect. 1.8.2 the only conditions that have to be satisfied are

$$\chi_f \frac{dT_f}{dz} = \chi_s \frac{dT_s}{dz}, \qquad T_f = T_s$$

where the index f refers to the fluid and the index s to the solid. However, the temperature field in the solid is not known and needs to be computed as well. The general case is thus quite tedious and we refer the reader to the work of Hurle et al. (1966) for a detailed study.

Here, we shall concentrate on the limit $\chi_s/\chi_f \to 0$ which is the case where the solid is a very poor heat conductor compared to the fluid. This case corresponds to the ideal insulator. Hence, after studying the ideal conductor case, we now explore the other extreme. From a physical point of view, it means that the temperature field in the solid is fixed (or evolve on a very long time scale compared to that of the fluid). Thus, the temperature gradient, and therefore the energy flux, is fixed in the solid and the temperature fluctuations at the interface do not propagate inside the solid. Hence, one imposes that the temperature gradient does not fluctuate, or that

$$\frac{\partial \theta}{\partial z} = 0 \tag{7.74}$$

on $z = 0, 1$. We note that in such conditions the Nusselt number remains fixed to unity.

The interesting point of this system is that the convective instability occurs with a vanishing critical wavenumber. Hurle et al. (1966) indeed noticed that as $\chi_s/\chi_f \to 0$ then $k_c \to 0$. Convection sets in at a scale all the larger that the solid is less conductive. It is then possible to find out a weakly nonlinear solution taking advantage of the fact that the horizontal scale is very large compared to the height. The resolution of this problem is a typical example of *a multi-scale analysis.*

7.

7.8.2 Formulation

We start again from the equation of motion (7.31) and introduce the small parameter ε such that:

$$\psi = \varepsilon\phi, \quad \partial_x = \varepsilon\partial_X, \quad \partial_t = \varepsilon^4\partial_\tau, \quad \mathrm{Ra} = \mathrm{Ra}_c + \mu^2\varepsilon^2$$

where μ measures the rate of supercriticality. Thus doing, we rescaled the horizontal lengths, introducing the scaled variable $X = \varepsilon x$ of order unity. We also rescaled the time and introduced the new time variable $\tau = \varepsilon^4 t$. Hence, we can focus on very large horizontal scales and very long time scales. The choice of the ε^4 factor in the time scale is justified a posteriori by the consistency of the solutions. The two equations of (7.31) now read:

$$\varepsilon^6\partial_\tau\partial_X^2\phi + \varepsilon^4\left(\partial_\tau D^2\phi + \partial_X\phi\partial_X^2 D\phi - \partial_X^3\phi D\phi\right) + \varepsilon^2\left(\partial_X\phi D^3\phi - \partial_x D^2\phi D\phi\right) =$$
$$\mathcal{P}\left[(Ra_c + \mu^2\varepsilon^2)\partial_X\theta + D^4\phi + 2\varepsilon^2\partial_X^2 D^2\phi + \varepsilon^4\partial_X^4\phi\right]$$

$$\varepsilon^4\partial_\tau\theta + \varepsilon^2\left(\partial_X\phi D\theta - \partial_x\theta D\phi\right) = \varepsilon^2\partial_X\phi + (D^2 + \varepsilon^2\partial_X^2)\theta$$

Here, the functions depends on the three variables (τ, X, z). The boundary conditions at $z = \pm\frac{1}{2}$ are[6]

$$D\theta = 0$$

for the temperature and

$$u_z = \varepsilon^2\partial_X\phi = 0 \qquad \text{and} \qquad \sigma_{xz} = 0 \Longleftrightarrow D^2\phi = 0$$

for the velocity. Note that we chose the stress-free boundary conditions; for no-slip conditions we would ask $u_x = 0$ or $D\phi = 0$.

7.8.3 The Chapman–Proctor Equation

We now develop the solution in powers of the small parameter up to the fourth order,

$$\theta = \theta_0 + \varepsilon^2\theta_2 + \varepsilon^4\theta_4 + \cdots, \qquad \phi = \phi_0 + \varepsilon^2\phi_2 + \varepsilon^4\phi_4 + \cdots$$

[6] We place the boundaries at $z = \pm\frac{1}{2}$ rather than at $z = 0, 1$ so as to be able to use the symmetry or the anti-symmetry of the functions with respect to the $z = 0$ plane.

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Note that with the choice made on the amplitudes and the horizontal scales, the velocity field is $\mathcal{O}(\varepsilon)$, or

$$\mathbf{u} = -\frac{\partial \psi}{\partial z}\mathbf{e}_x + \frac{\partial \psi}{\partial x}\mathbf{e}_z = -\varepsilon D\phi \mathbf{e}_x + \varepsilon^2 \partial_X \phi \mathbf{e}_z$$

whereas the temperature field remains $\mathcal{O}(1)$.

At zeroth order, the equations of motion reduce to

$$D^2\theta_0 = 0 \qquad \text{and} \qquad \mathrm{Ra}_c \partial_X \theta_0 + D^4\phi_0 = 0$$

which lead to the following type of solution

$$\theta_0 = f(X, \tau) \qquad \text{and} \qquad \phi_0 = \mathrm{Ra}_c P(z) f'(X, \tau)$$

where $D^4 P(z) = -1$. $f(X, \tau)$ is an unknown function which needs to be determined; $f'(X, \tau)$ is its derivative with respect to X. We note that the boundary conditions on θ_0 are automatically satisfied whereas those on the velocity demand that $P(\pm\frac{1}{2}) = 0$ and $P'(\pm\frac{1}{2}) = 0$ for no-slip conditions or $P''(\pm\frac{1}{2}) = 0$ for stress-free ones. These last two conditions and the differential equation allow us to specify completely the function $P(z)$. In the no-slip case

$$P(z) = -\frac{1}{24}z^4 + \frac{z^2}{48} - \frac{1}{384} = -\frac{1}{24}\left(z^2 - \frac{1}{4}\right)^2$$

while in the stress-free one

$$P(z) = -\frac{1}{24}z^4 + \frac{z^2}{16} - \frac{5}{384}\,.$$

Let us now consider the ε^2-order of the temperature equation. We have

$$D^2\theta_2 = -\mathrm{Ra}_c DPf'^2 - (\mathrm{Ra}_c P + 1)f'' \tag{7.75}$$

This equation is interesting as it has a solution only if the right-hand side verifies a solvability condition. Indeed, if we integrate the equation on z, then the left-hand side is zero whereas the right-hand side implies:

$$\mathrm{Ra}_c = -\left(\int_{-1/2}^{+1/2} P(z)dz\right)^{-1}$$

giving the value of the critical Rayleigh number. This expression leads to the numerical values Ra_c=720 in the no-slip case and Ra_c=120 in the free-slip one, values which were first derived by Hurle et al. (1966).

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The equation (7.75) can now be solved. We find

$$\theta_2 = f_2(X,\tau) + W(z)f'^2 + Q(z)f''$$

where we introduced $W(z)$ and $Q(z)$ such that

$$W'' + \mathrm{Ra}_c P' = 0 \qquad \text{and} \qquad Q'' + \mathrm{Ra}_c P + 1 = 0$$

These new functions verify the boundary conditions $Q'(\pm\frac{1}{2}) = W'(\pm\frac{1}{2}) = 0$ since $D\theta = 0$ on the boundaries. We infer that $W' = -\mathrm{Ra}_c P$.

The ε^2-order of the momentum equation leads to

$$D^4\phi_2 = -\mu^2 f' - \mathrm{Ra}_c\left[f_2' + W(f'^2)' + (Q+2P'')f'''\right] + \frac{\mathrm{Ra}_c^2}{\mathcal{P}}\left[PP''' - P'P''\right]f'f''$$

which is solved in the same way as the equation for θ_2; we find

$$\phi_2 = \mu^2 Pf' + \mathrm{Ra}_c Pf_2' + Uf''' + Sf'f''$$

with

$$D^4U = -\mathrm{Ra}_c(Q+2P'') \qquad \text{and} \qquad D^4S = -2\mathrm{Ra}_c W + \frac{\mathrm{Ra}_c^2}{\mathcal{P}}\left(PP''' - P'P''\right)$$

The boundary condition $u_z = 0$ imposes that

$$U(\pm 1/2) = S(\pm 1/2) = 0$$

The last step consists in writing the fourth order ε^4-term of the temperature equation. Integrating this equation on z between $\pm\frac{1}{2}$, we obtain

$$\partial_\tau f + A\mu^2 f'' + Bf^{(4)} + C\left(f'^3\right)' + D\left(f'f''\right)' = 0 \tag{7.76}$$

which is the *Chapman–Proctor equation*. It controls the horizontal dynamics of small-amplitude convection at fixed flux (Chapman and Proctor 1980). The constants A, B, C, D are given by

$$A = \frac{1}{\mathrm{Ra}_c}, \qquad B = -\int_{-1/2}^{1/2}(U+Q)dz,$$

$$C = -\mathrm{Ra}_c^2\int_{-1/2}^{1/2}P^2 dz, \qquad D = \int_{-1/2}^{1/2}(\mathrm{Ra}_c PQ' - S - 2W)dz$$

/ 10,

The evaluation of the foregoing integrals is not straightforward. Let us illustrate their derivation in the case of stress-free conditions on both boundaries. Because of the symmetry of the set-up $D = 0$. The calculation of B is a little tedious. We first remark that

$$-\int_{-1/2}^{1/2} U dz = \int_{-1/2}^{1/2} UD^4 P dz = \int_{-1/2}^{1/2} PD^4 U dz$$

Then, the differential equation verified by U implies that

$$B = \int_{-1/2}^{1/2} \left(2\mathrm{Ra}_c (DP)^2 - (DQ)^2\right) dz$$

Noting that

$$DQ = z^5 - \frac{5}{2}z^3 + \frac{9}{16}z$$

we finally obtain

$$B = \frac{1091}{5544} \simeq 0.197$$

In the same way, one can derive that

$$C = -\frac{155}{126}$$

7.8.4 *Properties of the Small-Amplitude Convection*

Chapman–Proctor's equation gives a good description of the dynamics when the temperature gradient is slightly supercritical.

To start with, let us examine the linear case and search for a solution proportional to $e^{\lambda t}$; if $\mu = 0$ (i.e. $\mathrm{Ra}=\mathrm{Ra}_c$), then the growth rate of a disturbance is just $-Bk^4$ and the critical wavenumber is $k = 0$ as expected. If the Rayleigh number is now slightly supercritical, we may linearize (7.76) and find the dispersion relation

$$\lambda = k^2 A\mu^2 - Bk^4 \tag{7.77}$$

which shows that the wavenumber of the fastest growing mode is

$$k_m = \frac{\mu}{\sqrt{2BRa_c}} \tag{7.78}$$

11.

This results shows that the fastest growing mode is not necessarily the one with the critical wavenumber. In the present case, the mode with the critical wavenumber ($k = 0$) has a zero growth rate!

We should also note that the transition from the hydrostatic state to the convective one is independent of the Prandtl number. The growth rate is real so that convection is steady (no oscillation).

Let us now examine the nonlinear régime. If the boundary conditions are identical on the top and bottom plates, the solution is symmetric with respect to the mid-layer $z = 0$ plane. The integrand defining D is antisymmetric and thus D should be zero in this case. The Chapman–Proctor equation therefore simplifies in this case and may be written

$$\partial_\tau g + A\mu^2 g'' + Bg^{(4)} + C\left(g^3\right)'' = 0$$

where we took the derivative of the equation and set $g = f'$. Now, introducing the new variable $u = \sqrt{\frac{C}{A}\frac{g}{\mu}}$ and changing the time scale as well as the X-scale, we find the Cahn–Hilliard equation:

$$u_t = -u'' - \mathcal{B}u^{(4)} + \left(u^3\right)'' \tag{7.79}$$

where $\mathcal{B} = BC^{1/2}/A^{5/2}\mu^5$. This equation was uncovered by John Cahn and John Hilliard in 1958 when they studied the dynamics of the phase separation phenomena.[7]

We note that this equation, as the Chapman–Proctor one, is richer than Landau equation which allowed us to study the nonlinear evolution of disturbances leading to convection rolls. The Landau equation indeed controls the time evolution of the amplitude of perturbations (whose structure is fixed by the linear analysis), while the two foregoing equations control both the time evolution and the spatial structure of the solutions (being partial differential equations). They are much simpler than the original ones, but still contain a rich variety of solutions. For instance, one can solve the Cahn–Hilliard equation in a stationary case (the solution is expressed with elliptic integrals) and then study the stability of these nonlinear solutions. Chapman and Proctor have shown that, in a periodic box, the stable flow is made of very flattened contra-rotating rolls.

[7] Cahn and Hilliard (1958).