

Convection Patterns I

So far:

- basis of dynamics (information)
 - dimension, attractors
 - Lyapunov exponents, chaos
- Patterns I - Phase Dynamics → Time:
 - single oscillator synchron, noise (CGL)
 - local coupling → phase diffusion
 - ↑ (Kuramoto) model → domains - f
 - ↓ synchron. PDF(ϕ)
 - global coupling (vs. dispersion, noise) → k transition → global synchron.
 - (nb: range of coupling is key)
 - phase turbulence, repulsive coupling → k-S eqn, etc.

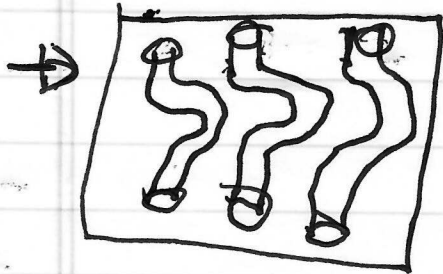
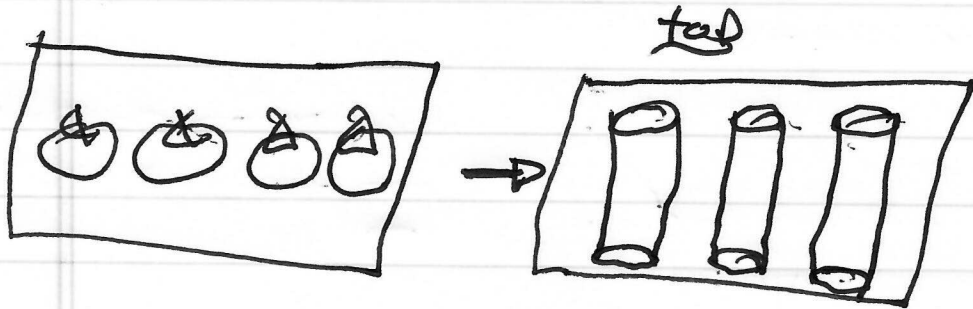
Now:

→ Patterns II - Convection near marginal
→ Space

Focus: Secondary instability in ensemble of convection cells/rolls near

marginality (i.e. 'weakly nonlinear'),

i.e.



Zig = Zag

i.e. development

- Eckhaus \rightarrow modulate array
- Zig-zag \rightarrow bending

- subject is classic, as easily amenable to experiment, tractable as (near equilibrium).

- approach:

- basic model of marginal roll \Rightarrow Swift-Hohenberg model.

- modulations in ensemble of rolls (some similarity to WKE).
i.e. pattern dynamics on scales

larger than that of individual roll...

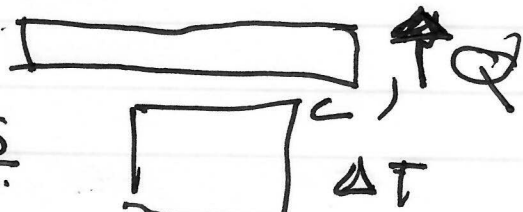
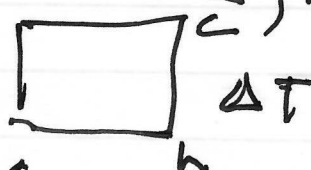
\Rightarrow envelope equations (Landau Theory)


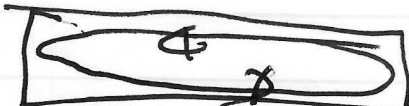
- implications of envelope formalism for patterns.

Comment:

- more elegant and 'classical' than useful. A must for a basic course. . . .

- related, more relevant models:

→ Fixed flux convection:  (Chapman - Proctor) vs 

→ convection driven flows:  → 

(Howard - Krishnamurti)

Refs:

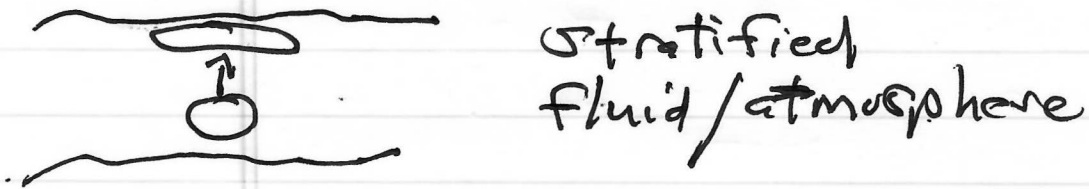
→ several posts → Cross + Hohenberg is encyclopedic.

→ books: $\left\{ \begin{array}{l} \text{Cross + Greenside} \\ \text{P. Manneville} \\ \text{Rebecca Hoyle} \end{array} \right.$

→ later: $\left\{ \begin{array}{l} \text{Chapman - Proctor} \\ \text{Howard, Krishnamurti} \end{array} \right\}$ Key papers.

→ Physics of Convection (Rayleigh-Bénard)

(See Lectures VI, Phys. 216, W 2017 or any book).



$$\underline{\underline{D \cdot \underline{v} = 0}} \quad \text{as } T > \ell / c_s;$$

→ parcel will rise by buoyancy if:

$$dS/dz < 0 \quad \Rightarrow \quad \frac{1}{T} \frac{dT}{dz} < \frac{(\gamma-1)}{\rho} \frac{d\rho}{dz} \quad (\text{geo})$$

$$\frac{1}{T_0} \approx \frac{\rho_0}{\gamma} \frac{dS_0}{dz}$$

→ if dissipation:

$$\left. \begin{aligned} \partial_t \tilde{T} &\rightarrow \partial_t \tilde{T} - \kappa \nabla^2 \tilde{T} \\ \partial_t \tilde{\underline{v}} &\rightarrow \partial_t \tilde{\underline{v}} - \nu \nabla^2 \tilde{\underline{v}} \end{aligned} \right\} \begin{array}{l} \text{viscosity,} \\ \text{heat diffusion} \\ \text{can damp} \\ \text{convection} \end{array}$$

so, natural to require for instability:

$$\frac{T_s T_x}{T_b^2} > 1 \Rightarrow g \frac{d\langle \zeta \rangle}{dz} l^4 / \nu \chi \equiv Ra$$

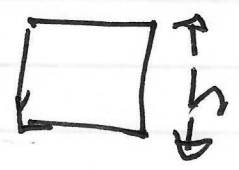
$$Ra > Ra_{crit.}$$

in box:

$$Ra \equiv \frac{g \Delta T \alpha h^3}{\nu \chi}$$

Rayleigh #

$\alpha = -\frac{1}{T} \frac{dT}{T}$
 coeff of thermal expansion



basic equations:

$$\partial_t \nabla_{\perp}^2 \phi - \nu \nabla_{\perp}^2 \nabla_{\perp}^2 \phi = -g \frac{d}{dz} \left(\frac{\tilde{T}}{T_0} \right) \quad (20)$$

$$\partial_t \left(\gamma \frac{\tilde{T}}{T_0} \right) = -\hat{v}_z \frac{d\langle \zeta \rangle}{dz} + \chi \nabla_{\perp}^2 T$$

$$\left(\underline{v} = \nabla \phi \times \hat{y} \right)$$

or (with $w = v_z$)

$$\frac{\partial}{\partial t} \nabla^2 W = g\alpha \left(\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} \right) + \nu \nabla^2 \nabla^2 W \quad (6)$$

$$\frac{\partial \Theta}{\partial t} = \beta W + \kappa \nabla^2 \Theta$$

What sets (behavior of) critical Rayleigh # \rightarrow dissipation, and boundary conditions

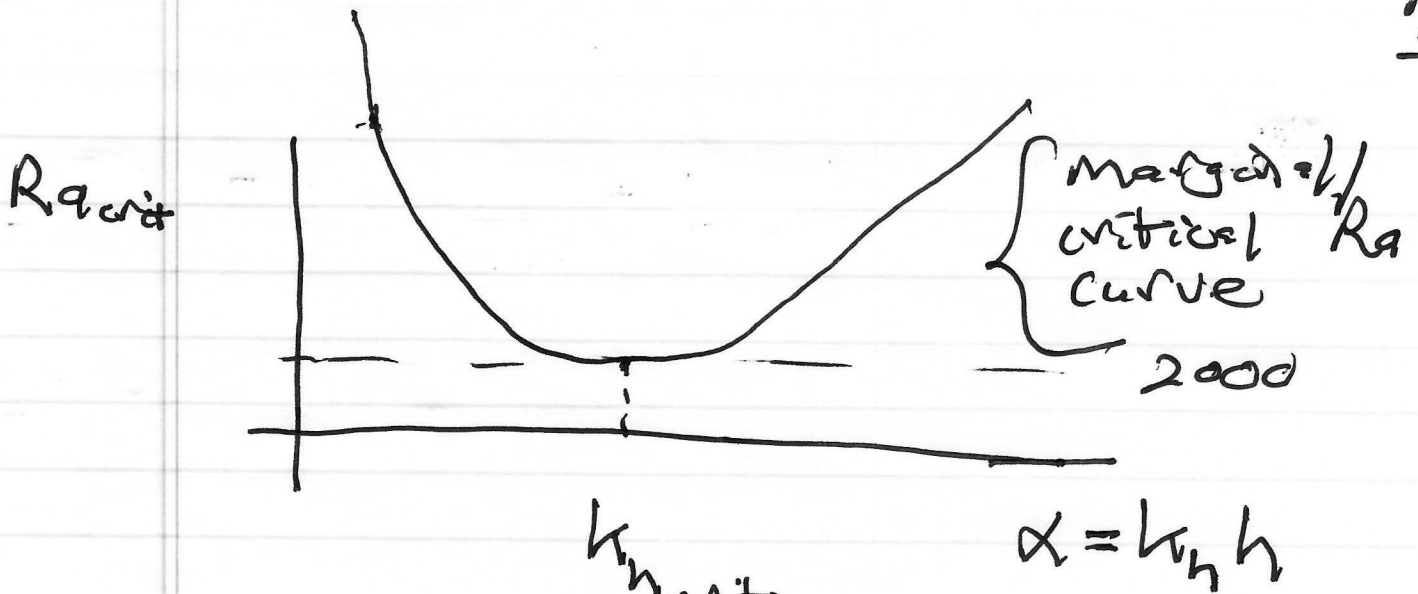
ie for no slip:

$$\tilde{V}_z \Big|_{0,h} = 0 \quad ; \quad \tilde{V}_h \Big|_{0,h} = 0$$

n.b.: ~~∇_h~~ $\nabla_h \tilde{V}_h + \nabla_z \tilde{V}_z = 0$

no slip $\Rightarrow \nabla_h \tilde{V}_h = 0$

so $\partial_z W \Big|_{0,h} = 0$



→ high k_n
 ~ $\sim k^2$, etc

→ minimum Ra_{crit} .

→ low k_n



~ no slip ($U_n = 0$) damping.

N.B.: In stress free, numbers change, but similar structure.

→ Now, how describe convection for

$$Ra = Ra_{crit} + \sum \delta Ra$$

i.e. small excursion into super-critical?

ID Key elements:

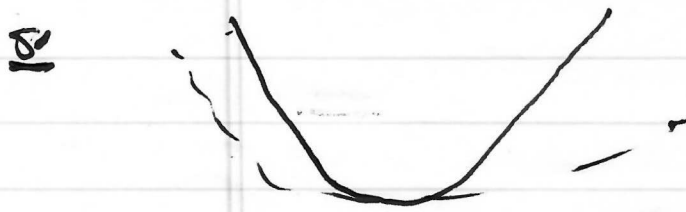
- Ra_{crit}

- k_{crit}

→  curve

→ saturation

~ system approaches steady state.



fit with parabola.

$$\delta \tau_0 = (Ra - Ra_{crit}) - \tau_0 \Sigma^2 (k - k_{crit})^2$$

model of growth near marginal.

∞, schematically:

$$\tau_0 \frac{\partial W}{\partial t} = (Ra - Ra_{crit}) W - \tau_0 \Sigma^2 (\sqrt{-\alpha_x^2} - k_{crit})^2 W - |W|^2 W$$

has form of Landau Eqn.

How simplify?

This is a step toward Swift-Hohenberg Model a reduced model of

convection near onset. See Swift, Hohenberg 1977 for derivation.

Now, beyond 1D; consider: (de-10)

- uniform base state
- rotationally invariant in 2D plane (⊥ rolls)

so $\gamma_{\underline{q}}$ can depend only on $|\underline{q}| = \underline{q}$ not \underline{q} .

⇒ control parameter ($Ra - Ra_{crit}$)

$$\gamma_{\underline{q}} = \rho - c(\underline{q} - \underline{q}_0)^2$$

↓
min \underline{q}
growth

⇒ $\partial_t W = \left(\rho - c(\sqrt{1-D^2} - \underline{q}_0)^2 \right) W + \dots$

Now, near onset:

$z + z_c \approx 2z_c$, so can write 1
in a "creative way":

$$\begin{aligned} \gamma &= \rho - c(z - z_c)^2 \\ &\approx \rho - c \frac{(z + z_c)^2 (z - z_c)^2}{(2z_c)^2} \end{aligned}$$

$$\approx \rho - \frac{c}{4z_c^2} (z^2 - z_c^2)^2$$

\Rightarrow re-scale:

$$\partial_t W = rW - (D^2 + z_c^2) W$$

frequently $\rightarrow 1$

Finally, to saturate, need restrict growth at finite amplitude and:

- respect chiral symmetry
 $w \rightarrow -w$ (cf. basic eqns)
- respect phase symmetry.

$\Rightarrow -W^3, -|W|^2 W$

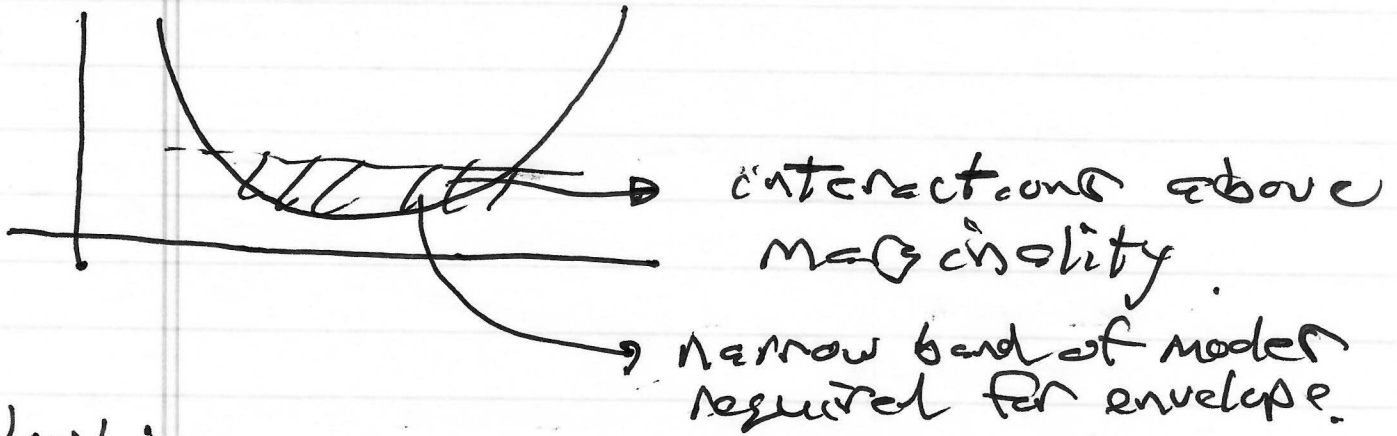
$\partial_t W = rW - (D^2 + z_0^2)^2 W - |W|^2 W$

Swift - Hohenberg Model

\rightarrow describes convection near onset

\rightarrow can't quantify its own breakdown.

i.e. describes:



Now:

- S-H can be derived from basic eqns. systematically.
Tedious, not instructive.

→ $\mathcal{J}-H$ is derivable from variational principle - i.e. Lyapunov Function

Now, for W^3 form:

$$V[W] = \int dx \int dy \left[-\frac{1}{2} r W^2 + \frac{1}{4} W^4 + \frac{1}{2} [(\nabla^2 + 1)W]^2 \right]$$

and

$$\frac{dV}{dt} = - \int dx \int dy (\partial_t W)^2$$

i.e., any evolution of W tends to

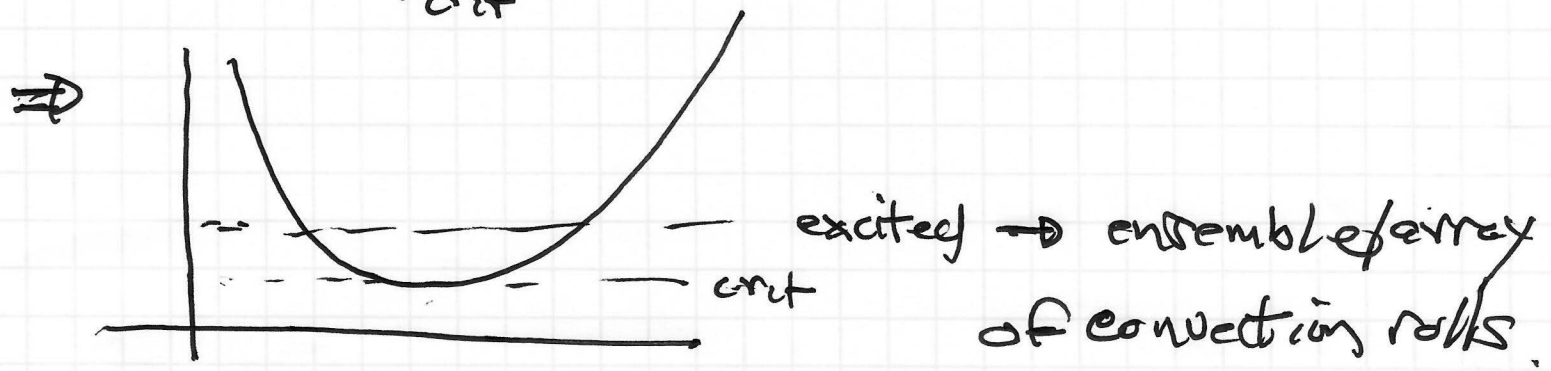
decrease V . V is minimum at stationarity of W .

→ easily shown.

→ can write: $\partial_t W = - \frac{\delta V}{\delta W}$

→ J-H Model is a reduced model, applicable to band of modes near Ra_{crit} .

→ $Ra = Ra_{crit} + \delta Ra$



what happens? → 132.

With models:

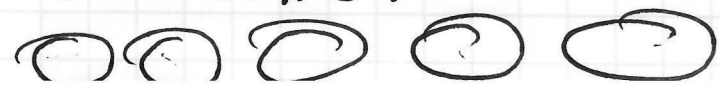
key } how does pattern of excited cell evolve? what configuration does it adopt.

Classif:

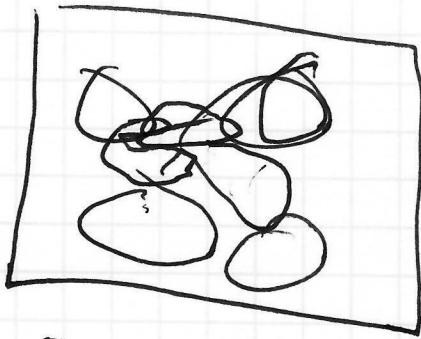
⇒ explore stability of band of modes with: breaks symmetry

$\underline{z} = z_0 \hat{x} + \underline{h}$

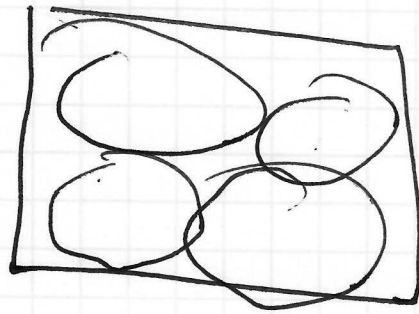
$- z_0 \hat{x} \rightarrow$ base state is array of cells:



→ See 184-185 of Cross, Greenside Ba.



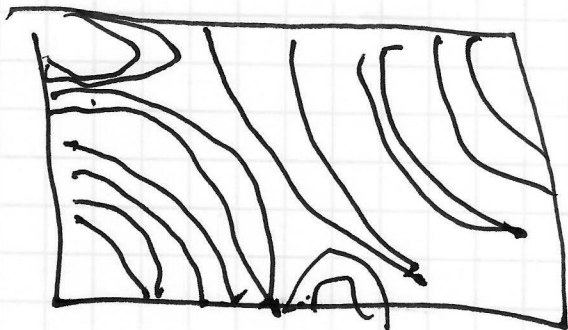
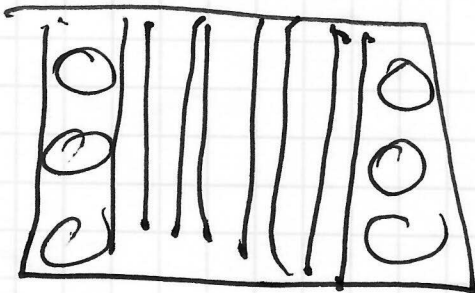
domains, periodic
box



larger
domains
→ coarsening

⇒ Eckhaus
process.

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- $|k|/|z_0| \ll 1$, so

$$W \equiv \left[W_0 A(x, y) e^{i z_0 x} + c.c. \right] + o(\epsilon)$$

amplitude carrier

Now, backing up:

$$\gamma T_0 = \epsilon - \underbrace{\Sigma_0^2}_{\text{Ra-Rcart}} (z - z_0)^2$$

Now, $\underline{z} = \underbrace{z_0 \hat{x}}_{\substack{\text{1D} \\ \text{base}}} + \underbrace{k}_{\text{2D}}$

so

$$\begin{aligned} \gamma T_0 &= \epsilon - \Sigma_0^2 \left(|z_0 \hat{x} + k| - z_0 \right)^2 \\ &= \epsilon - \Sigma_0^2 \left(z_0 \left(\left(1 + \frac{k_x^2}{z_0^2} + \frac{k_y^2}{z_0^2}\right)^{1/2} - z_0 \right) \right)^2 \end{aligned}$$

expanding, etc.

$$\delta \mathcal{L}_0 = \epsilon - \epsilon_0^2 \left(k_x + \frac{k_y^2}{2\epsilon_0} \right)^2$$

↓
 envelope \mathcal{H}
 dependence

h.o. on
 k_x coord.

Note:

- k_x, k_y asymmetry due to symmetry breaking / direction set of the base state

→ envelope eqn!

$$\partial_t W = \epsilon - \epsilon_0^2 (z - z_0)^2$$

$$= \epsilon W - \epsilon_0^2 \left(\left[\frac{z_0 + \mathcal{H}}{\epsilon_0} \right] - |z_0| \right)^2 W$$

$$W = w_0 A e^{i \mathcal{L}_0 X}$$

→ op.

$$\mathcal{H} \rightarrow \partial_x A$$

$$\mathcal{L} \rightarrow \partial_t A$$

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$$\gamma \rightarrow \partial_t$$

$$k_x \rightarrow -i \partial_x$$

$$k_y \rightarrow -i \partial_y$$

16.

and: $\omega \rightarrow -\omega \Rightarrow \omega^3$ saturation

$\omega \rightarrow \omega e^{i\phi} \Rightarrow$ spatial shift of pattern

irrelevant

(boundaries)

\Rightarrow Newell - Whithead Eqn. (Amplitude)

$$\partial_t A = A + \left(\partial_x - \frac{g}{2} i \partial_y^2 \right)^2 A - |A|^2 A$$

where:

re-scale

$$x = |E|^{1/2} x / \epsilon_0$$

$$y = |E|^{1/4} y (g_0 / \epsilon_0)^{1/2}$$

$$T = \epsilon t / \tau_0$$

$$A = (g_0 / |E|)^{1/2} A.$$

→ NW has Lyapunov Function

17.

→ Difference of $\begin{cases} \widetilde{SH} \rightarrow \text{maintains rotational symmetry base state} \\ \widetilde{NW} \rightarrow \text{breaks symmetry} \end{cases}$
by $\underline{z} = z_0 \hat{x}$
assumption.

Now, have N-W eqn:

$$\gamma_0 \partial_t A = r A + \epsilon_0^2 \left(\partial_x + \frac{1}{2ik_c} \partial_x^3 \right)^2 A - g |A|^2 A$$

(CKM CGL)

and useful to re-write as:

$$A = |A| e^{i\phi} \rightarrow q e^{i\phi}$$

as complex structure ~~guarantees~~ phase dynamics relevant:

ignoring y dep,

$$\gamma_0 \partial_t q = \left(r - \epsilon_0^2 (\partial_x \phi)^2 \right) q + \epsilon_0^2 \partial_x^3 q - g |q|^2 q$$

amplitude.

$\overset{b}{a^3}$

$$\partial_t \phi = \frac{\epsilon_0^3}{\hbar^3} \left(\partial_x^2 \phi + 2 \frac{\partial_x a}{a} \partial_x \phi \right) \frac{1}{\epsilon_0}$$

phase

observe:

- amplitude evolves via r and q^3 , favoring long wavelengths
- phase diffusive. Can $2 \frac{\partial_x a}{a} \partial_x \phi$ (effectively) flip sign of effective diffusivity?

Time independent solutions:

→ phase winding

$$\partial_x a = 0$$

$$\phi = \delta k x + \phi_0$$

$$\Rightarrow 0 = (r - \epsilon_0^2 \delta k^2) a - g a^3$$

$$a = \left[(r - \epsilon_0^2 \delta k^2) / g \right]^{1/2}$$

$$\Rightarrow A = e^{i \delta k x + \phi_0} a$$