Convection Patterns I, cont'd (again). 1

Phase Dynamics

so far:

i.) are concerned with patterns near primary instability onset

ii.) primary linear stability properties (+ minimal symmetry assumption)

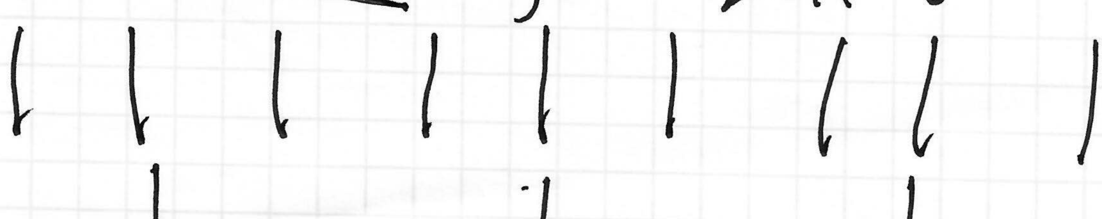
i.e. $$\gamma \tau_0 = (Ra - Ra_{crit}) - \xi_0^2 (k - k_0)^2$$

selects (possible) pattern base state

iii.) seek determine "textures" on base state [via envelope formalism] → secondary "roughening"

cf. Pattern-phasing $\Leftrightarrow$ textures – phasing

example: Eckhaus $\lambda_{eff} \uparrow$ $k_{eff} \downarrow$



$\Rightarrow$

Now,

– recall "phase winding solution" effectively slaves amplitude to phase, i.e. $(\partial_y = 0)$

$$\partial_t \phi = \frac{\varepsilon_0^2}{\tau_0}\left(\partial_x^2 \phi + 2\frac{\partial_x |A| \, \partial_x \phi}{|A|}\right)$$

$$\tau_0 \partial_t |A| = \left(r - \varepsilon_0^2 (\partial_x \phi)^2\right)|A| + \varepsilon_0^2 \partial_x^2 |A| - g_{eff} |A|^3$$

$$\phi = \delta k \, x + \tilde{\phi}$$

$\varepsilon_0^2 \delta k^2 \sim r \Rightarrow$ eliminates / reduces amplitude growth

and observe, too:

$$\partial_t \phi \cong \frac{\varepsilon_0^2}{\tau_0} \partial_x^2 \phi + \cdots$$

– diffusive

– slow at large scales.

⇒ The Point:

Textures $\begin{cases} \text{instability} \\ \text{noise driven} \end{cases}$ $\Leftrightarrow$ $\begin{cases} \text{rooted to} \\ \text{phase evolution} \end{cases}$ {i.e. δk

Can one exploit long wavelength ordering to look beyond linear perturbation theory (basic Eckhaus / Zig-Zag theory is perturbative - linear) to examine pattern formation. Phasing is place to look.

Is there something deeper ??

yes → look at slowly varying phase parts ∂

⇒ Phase Diffusion Formalism

(c.f. Pomeau + Manneville, '79)

- key idea: invariance under const phase shift ⟹ approximate invariance under weakly varying phase shift

⇒ small error;

- nonlinear eikonal theory

- seek:

$$\partial_t \phi = D_{\parallel} \partial_x^2 \phi + D_{\perp} \partial_y^2 \phi + \cdots$$

what are $D_{\parallel}$, $D_{\perp}$? (when < 0).

— base state is <u>nonlinear</u>.

Proceeding:

— from SH model: (already several assumptions)

$$\partial_t W = rW - (\partial_x^2+1)^2 W - W^3$$

$$\partial_t W = 0 = rW - (\partial_x^2+1)^2 W - W^3$$

$r > 0 \Rightarrow |\delta k| \neq 0$. phase winding, $\varepsilon_0 \delta k \ll 1$

now $|\delta k| \leq \sqrt{r}/2$ $\rightarrow$ $\Big\{$ primary unstable band

and <u>periodic</u> stationary solutions:

$$W_0(x) = W_1 \sin(q_0 x) + W_3 \sin(3q_0 x)$$

where: $W_1 = \left(\frac{4}{3}(r - 4\delta k^2)\right)^{1/2}$

base state stationary exact soln $\Big\}$ $W_3 = W_1^2/256$

— Now, first consider <u>uniform</u> translation

$W_0(x)$ solution $\Rightarrow$ $W_0(x+\phi)$ solution so

$$w_0(x+\phi) = w_0(x) + \phi\,\partial_x w_0(x) + \frac{\phi^2}{2}\partial_x^2 w_0(x) + \cdots$$

$\Rightarrow$

$$\partial_t\left(w_0 + \phi\,\partial_x w_0 + \cdots\right)$$

$$= 0 = F\left(w_0 + \phi\,\partial_x w_0 + \cdots\right)$$

RHS, S-H operator

$$= F(w_0) + \left.\frac{\delta F}{\delta w}\right|_{w_0} (\phi\,\partial_x w_0) + \cdots$$

$$\Updownarrow$$

$$\Lambda_0.$$

i.e.

$$\begin{cases} F(w_0) = r w_0 - (\partial_x^2+1)^2 w_0 - w_0^3 \\ \underbrace{\Lambda_0}_{\text{opr.}} = r - (\partial_x^2+1)^2 - 3 w_0^2. \end{cases}$$

Now, by defn;

$$F(w_0) = 0 \quad , \text{ sol}^n.$$

so

$$\frac{\partial}{\partial x} F(W_0) = 0 = \frac{\partial F}{\partial W}(\partial_x W_0)$$

$$= \Lambda_0 (\partial_x W_0) \qquad \text{(operation)}$$

so

$$\partial_t (W_0 + \phi\, \partial_x W_0) = 0 = F(W_0) + \Lambda_0(\phi\, \partial_x W_0)$$

$$= F(W_0) + \phi \Lambda_0 (\partial_x W_0)$$

so as ϕ uniform

$$\boxed{(\partial_t \phi)\, \partial_x W_0 = \phi \Lambda_0 (\partial_x W_0) = 0}$$

i.e. – uniform phase does ~~not~~ evolve.

→ $\partial_x W_0$ neutral (i.e. translation mode)

so

– $\Lambda_0 |\psi\rangle = \lambda |\psi\rangle$

[redacted] $\langle x|\psi\rangle = \partial_x W_0$ [redacted] $+ \ldots$

i.e $\partial_x W_0$ eigenstate / belongs to kernel of Λ_0 operator

Now, allow for phase variation

i.e. non-uniform:

$\phi = \phi(x, y, t)$ $\Rightarrow$ {accomodates phase clustering

so

- $w(x+\phi)$ is not exact solution
- but error small in $\partial_x \phi$, etc.

i.e. phase shift is approaching exact solution as $\partial_x \phi \to 0$

i.e. long wavelengths.

so, now look for solutions of form:

exact for ϕ uniform

- $w(x, y, t) = w_0(x) + \phi\, \partial_x w_0 + w_1 + w_2$

$\phi\, \partial_x w_0 \to w_1$, $O(\epsilon)$

corrections for non-uniform ϕ

$w_2 \to O(\epsilon^2)$

- $\Delta\,(\phi\, \partial_x w_0)$

acts on both ϕ, $\partial_x w_0$.

so

$$\partial_t \left(W_0 + \phi\, \partial_x W_0 + W_1 \right)$$

$$= F(W_0) + A\left(\phi\, \partial_x W_0 + W_1\right)$$

$\Rightarrow$ (see 8a)

$$\begin{cases} A_0 W_1 = \partial_t \phi\, \partial_x W_0 + g(x)\, \partial_x \phi \\ g(x) = 4\left(\partial_x^2 + 1\right) \partial_x^2 W_0 \end{cases}$$

<u>seek ϕ</u>

$\boxed{\text{somewhat akin Chapman-Enskog}}$

Fredholm alternative replaces $C(\epsilon) = 0$ (solvability).

Now,

- $\partial_x W_0$ appears on RHS
- $\partial_x W_0$ is eigenmode of A_0 (i.e. in <u>kernel</u>)

Fredholm Alternative: (avoid $\frac{0}{0}$)

- Can solve for W_1 <u>only</u> if RHS $\perp$ [kernel $A_0^{\dagger}$]

$\perp$: orthogonal; $\dagger \to$ adjoint

$\rightsquigarrow$ solvability condition

$-\Lambda(\phi\,\partial_x W_0) \to J(\alpha)\,\partial_x\phi$

via $\quad (\partial_x^2+1)^2(\partial_x W_0\,\phi)$

$\to (\partial_x^2+1)(\partial_x^2+1)\,\partial_x W\,\phi$

$\to (\partial_x^2+1)\,\partial_x^2 W\,\partial_x\phi$

but;

$$\Lambda_0 = \Lambda_0^{+} \quad \text{(self-adjoint)}$$

so RHS $\perp$ kernel Λ_0 required.

<u>but</u> kernel Λ_0 contains $\partial_x W_0$!

$\Rightarrow$

$$\langle \partial_x W_0 | \partial_t \phi \, \partial_x W_0 \rangle + \cancel{\langle \partial_x W_0 | \partial_x \phi \, g(x) \rangle}$$

odd deriv.

even deriv.

$= 0$

$\partial_t \phi = 0$, lowest order.

so, cannot treat $\phi \, \partial_x W_0$ as fast varying, only.

i.e. $\{W_0$ fast vel. $\{$modes

$\therefore$ [must isolate piece of W_1
[explicitly proportional $\partial_x \phi$.

<u>i.e</u>

$$W_1 = W_1^{(0)} + \partial_x \phi \, \tilde{W}_1(x)$$

$$= W_1^{(0)} + \partial_x \phi \, \tilde{W}_1(x)$$

10

$$\Lambda_0\left(w_1^{(0)} + \partial_x\phi\, \tilde{w}_1(x)\right)$$

$$\doteq \partial_t\phi\, \partial_x W_0 + g(x)\, \partial_x\phi$$

$$\cancel{\Lambda_0(w_1^{(0)})} + \underline{\partial_x\phi}\, \Lambda_0(\tilde{w}_1) = \cancel{\partial_t\phi\, \partial_x W_0}$$

$$+ g(x)\, \underline{\partial_x\phi}$$

homogeneous

$$\left\{ \begin{array}{l} \Lambda_0(\tilde{w}_1) = g(x) = 4\left((\partial_x^2+1)\right)\partial_x W_0 \\ \underline{\text{and crank for } \tilde{w}_1} \end{array} \right.$$

✓.

Now, expand to <u>second</u> <u>order</u> in <u>phase</u> <u>slope</u>:

$$\partial_x W_0\, \partial_t\phi + \partial_t W_1 + \cancel{\partial_t W_2} = 0$$

h.c.

$$= \Lambda_0\left(\phi\, \partial_x W_0 + \partial_x\phi\, \tilde{w}_1 + w_1 + W_2\right)$$

recall eq.

<u>so</u>

<u>crank</u> $\Rightarrow$

c.f. $\left\{ \begin{array}{l} \text{Hoyle} \\ \text{or} \\ \text{Manneville} \end{array} \right.$

1a

$$\left\{\begin{aligned} L_0 W_2 &= \partial_t\phi\, \partial_x W_0 + \partial_x^2\phi \Big[4(\partial_x^2+1)\partial_x \tilde{W}_1 \\ &\quad + 2(3\partial_x^2+1)\partial_x W_0\Big] \\ &\quad + \partial_y^2\phi \Big[2(\partial_x^2+1)\partial_x W_0\Big] \end{aligned}\right.$$

if retain full D_h

$\langle \partial_x W_0 \,|\, RHS \rangle = 0 \Rightarrow$ [solvability eqn.]

$$\boxed{\partial_t \phi = D_{\parallel}\, \partial_x^2 \phi + D_{\perp}\, \partial_y^2 \phi}$$

phase diffusion equation

$$\left\{\begin{aligned} D_{\parallel} &= -\Big\langle \partial_x W_0 \Big| \Big[2(3\partial_x^2+1)\partial_x W_0 \\ &\qquad + 4(\partial_x^2+1)\partial_x \tilde{W}_1\Big]\Big\rangle \\ &\quad * 1/\langle \partial_x W_0 | \partial_x W_0 \rangle \\ D_{\perp} &= -\langle \partial_x W_0 | 2(\partial_x^2+1)\partial_x W_0 \rangle * \\ &\quad 1/\langle \partial_x W_0 | \partial_x W_0 \rangle \end{aligned}\right.$$

and using ω_0, $\tilde{\omega}_1$:

$$\begin{cases} D_{\parallel} = 4\left[\dfrac{r-12\delta k^2}{r-4\delta k^2}\right] \\ D_{\perp} = \left(\delta k - \dfrac{(r-4\delta k^2)^2}{1024}\right) \end{cases}$$

$D_{\parallel} < 0 \Rightarrow$ Eckhaus (see previous)
interval: $\begin{cases} r-12\delta k^2 < 0 \\ r-4\delta k^2 > 0 \end{cases}$

$D_{\perp} < 0 \Rightarrow$ zig-zag.

What is new?

δk and correction $r^2/1024$.
$\begin{cases} \delta k > 0 \text{ but} \\ \text{tiny} \ldots \end{cases}$

$\Rightarrow$ derived classic secondary instabilities via $\begin{cases}\text{structural} \\ \text{approach}\end{cases}$

$\rightarrow$ scalar products involving NL solutions above threshold.

$\Rightarrow$ non-perturbative $\Rightarrow$ chain NL
eikonal theory.

$\Rightarrow$ observe (admittedly small) difference in zig-zag threshold

→ Now, can argue structure of phase equation from form:

$$\partial_t \phi = RHS$$

where:

- RHS function of phase curvature, gradients etc. Phase shift magnitude should have no impact.

- $W(x)$ soln → $W(x+\phi)$ soln.

so eqn. invariant under

$\phi \to -\phi$ $\qquad\qquad$ $y \to -y$

$x \to -x$.

so

$$\partial_t \phi = \cancel{c_1 \frac{\partial \phi}{\partial x}} + \cancel{c_2 \frac{\partial \phi}{\partial y}}$$

but can't extract form. D

$$+ D_\parallel \frac{\partial^2 \phi}{\partial x^2} + D_\perp \frac{\partial^2 \phi}{\partial y^2}$$

$$+ \cancel{\text{cubic}} - K \frac{\partial^4 \phi}{\partial x^2 \partial x^2}$$

$$+ J \left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial^2 \phi}{\partial x^2}\right) + \cdots$$

$$\partial_t \phi = D_{\parallel} \frac{\partial^2 \phi}{\partial x^2} + D_\perp \frac{\partial^2 \phi}{\partial y^2}$$

$$-K \frac{\partial^4 \phi}{\partial x^4} + g\left(\frac{\partial \phi}{\partial x}\right) \frac{\partial^2 \phi}{\partial x^2}$$

if $D_{\parallel} < 0$ → akin ~~[illegible]~~ K-S
modified

How obtain g ?

need $K > 0$
$D_{\parallel} < 0$ → Eckhaus

if ignore y (Eckhaus): re-scale

$$\boxed{\partial_t \phi = D_{\parallel} \partial_x^2 \phi - K_x \partial_x^4 \phi + g\, \partial_x \phi\, \partial_x^2 \phi}$$

$$\phi = \delta k / k + \tilde{\phi}$$

$\tilde{\phi}$ as phase for structure with
$k \to k + \delta k$.

$$\partial_t \tilde{\phi} = D_{\parallel} \partial_x^2 \tilde{\phi} - K \partial_x^4 \tilde{\phi} + \frac{g\,\delta k}{k} \frac{\partial^2 \tilde{\phi}}{\partial x^2}$$
$$+ g \frac{\partial \tilde{\phi}}{\partial x} \frac{\partial^2 \tilde{\phi}}{\partial x^2} \quad \text{h.o.}$$

$$D_{11}(k+\delta k) = D_{11}(k) + \delta k \frac{dD_{11}}{dk}$$

$$= D_{11}(k) + \frac{g}{k}\delta k$$

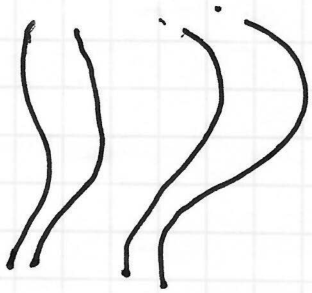
$\Rightarrow$

$$\boxed{g = k\, dD_{11}/dk}$$

→ Again asking re: what has all this bought us?

⇒ New way of looking at textures.

→ Consider constant phase contours of pattern:



i.e. = curvature

— dilatation

⇒ suggests director field!

i.e. $\underline{k} = k\,\hat{n}$

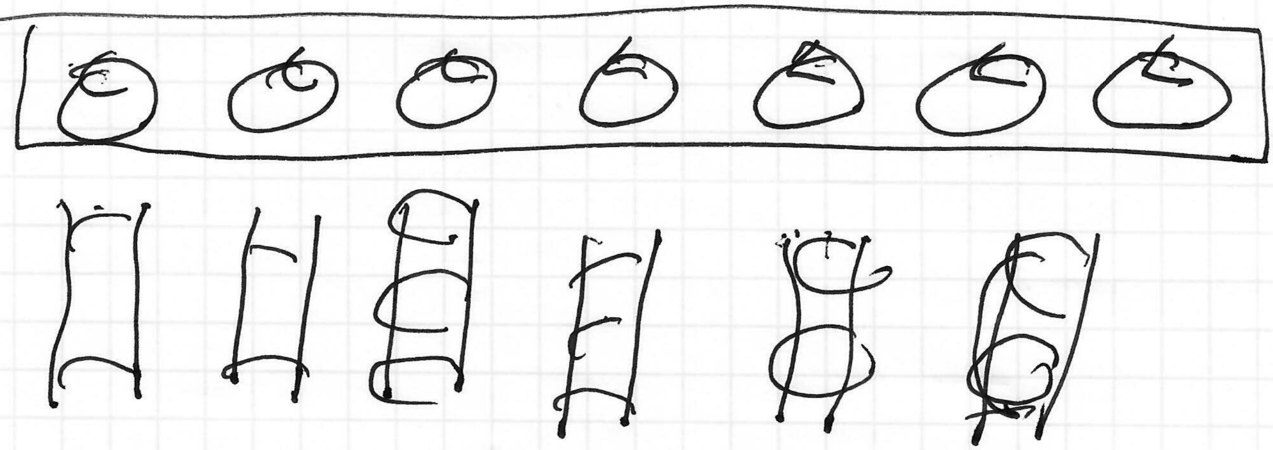
roll direction

scale

phase gradient

⇒ eikonal eqn.

i.e. consider phase, more generally



pattern → specified by level lines W

i.e. $W(x, y, t)$

or

$\equiv$ has period $2\pi/z_0$

$$V(x, y, z, t) = V_0\left(u(x, y, t), z\right)$$

solution — generalized phase describing horizontal dependence

have taken:

$$u = x + \phi(x, y, t)$$

, tacitly assumed ϕ small.

Now, with curvature change, etc., ϕ may not be small.

$\Rightarrow$ a la eikonal theory, better to to track phase gradient, i.e. $\underline{k}$.

Recall $\quad \dfrac{d\underline{k}}{dt} = -\dfrac{\partial}{\partial \underline{x}}(\omega) \qquad \underline{k} = \nabla \phi$

$$= -\frac{\partial}{\partial \underline{x}}\left(\omega + \underline{k}\cdot\underline{V}\right)$$

so, specify pattern by:

$$\underline{k}(x,y,t), \text{ with } \begin{cases} \underline{k} = \nabla_h \psi \\ \nabla \times \underline{k} = 0 \end{cases}$$

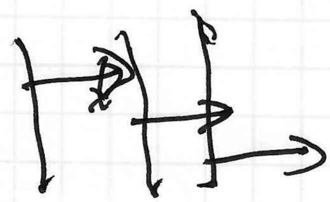
- ties $\underline{k}$ → phase gradient
- applies if periodic structure identifiable.

Now, $\underline{k}$ has direction, magnitude

so:

→ director

$$\underline{k} = k\,\hat{n}$$



k → wave number

local curvature

$$\nabla_h \cdot \underline{k} = \nabla_h \cdot (k\hat{n}) = k\,\nabla\cdot\hat{n} + \hat{n}\cdot\nabla_h k$$

$\hat{n}\cdot\nabla_h k$ → dilatation

Now can understand pattern dynamics in terms iso-phase lines

so

contour (phase) velocity !

$$\frac{\partial u}{\partial t} + V_n \hat{n}\cdot\underline{\nabla} u = 0 \qquad V_n \to \text{rot inv. only.}$$

and

$$\underline{V_n} = -D_{\parallel}\,\hat{n}\cdot\underline{\nabla}_n (k/k_0) - D_{\perp}\,\underline{\nabla}_n\cdot\hat{n}$$

check:

$$\frac{\partial u}{\partial t} - \left(D_{\parallel}\,\hat{n}\cdot\nabla_n (k/k_0)\right)\left(\hat{n}\cdot\underline{\nabla} u\right)$$

$$-\left(D_{\perp}\,\underline{\nabla}_n\cdot\hat{n}\right)\left(\hat{n}\cdot\nabla u\right) = 0$$

$$u = x + \phi$$

$$\partial_x u = 1 + \partial_x \phi = k_x/k_0 \qquad \underline{n} = (\cancel{1+\partial_x\phi}, \partial_y\phi)$$

$$\Rightarrow k/k_0 \cong 1 + \partial_x \phi$$

$$\Rightarrow \hat{n}\cdot\underline{\nabla} u \cong 1 +$$

$$\hat{n}\cdot\underline{\nabla}(k/k_0) = \partial_x^2 \phi$$

$$\Rightarrow \frac{\partial \phi}{\partial t} - D_{\parallel}\,\partial_x^2 \phi - D_{\perp}\,\partial_y^2 \phi = 0$$

→ recovers phase diffusion eqn.

i.e. $\quad u = x + \phi$

$$\frac{\partial u}{\partial t} + V_n \hat{n} \cdot \nabla u = 0$$

$$\underline{V_n} = - D_{\parallel} \hat{n} \cdot \nabla_n (k/k_0) - D_{\perp} \underline{\nabla_n} \cdot \hat{n}$$

→

$$\frac{\partial \phi}{\partial t} = D_{\parallel} \partial_x^2 \phi + D_{\perp} \partial_y^2 \phi$$

and can convert problem into phase dynamics equation:

$$\boxed{T(k) \partial_t \tilde{u} + \underline{\nabla_n} \cdot (\underline{k} \, B(k)) = 0}$$

$\tilde{u} = k u$ $\qquad \left.\begin{matrix} T(k) \\ B(k) \end{matrix}\right\}$ TBD

$$\tilde{u} = k(x + \phi)$$

$$T \partial_t \phi + \underline{\nabla_n} \cdot (\underline{k} \, B(k)) = 0$$

$$\underline{k} = k \hat{n}$$

$$T\partial_t\phi + \nabla_h\cdot(k\hat{n}\,B(k)) = 0$$

$$T\partial_t\phi + kB(k)\,\nabla_h\cdot\hat{n} + \frac{d}{dk}(kB(k))\,\hat{n}\cdot\nabla k = 0$$

from before

$$T\partial_t\phi + B(k)\,\partial_y^2\phi + \frac{d}{dk}(kB(k))\,\partial_x^2\phi = 0$$

so

$$D_{\parallel} = \frac{-1}{T}\frac{d}{dk}(kB(k))$$

$$D_{\perp} = \frac{-1}{T}B(k)$$

$$\frac{B(k)}{D_{\perp}} = \frac{1}{D_{\parallel}}\frac{d}{dk}(kB(k))$$

$$= \frac{1}{D_{\parallel}}\left(B(k) + k\frac{dB}{dk}\right)$$

$$\Rightarrow \boxed{\frac{dB}{B} = \frac{dk}{k}\left(\frac{D_{\parallel}}{D_{\perp}} - 1\right)}$$

concludes patterns in (standard) convection.