B.) Kuramoto Transition / Synchronization with Noise

Now:

- uniform frequency distribution (i.e. $g(\omega) = \delta(\omega - \omega_0)$)

but – noise !

Obviously:
- coupling favors synchronization
- noise scatters phases → opposes it.

single frequency

$$\frac{d\phi_K}{dt} = \omega_0 + \frac{\epsilon}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_K) + \underbrace{\mathcal{E}_K(t)}_{\text{noise}}$$

now: $\psi_K = \phi_K - \omega_0 t$

$$\langle \mathcal{E} \rangle = 0$$

$$\langle \mathcal{E}_m(t)\, \mathcal{E}_n(t') \rangle = 2\sigma^2 \delta_{m,n}\, \delta(t - t')$$

independent noises, each oscillator.

Competition clearly one of noise vs coupling, rather than dispersion vs. coupling.

As before, define mean field

$$Z = X + iY = \frac{1}{N} \sum_{k=1}^{N} e^{i\psi_k}$$

$$= \int_0^{2\pi} d\psi \, e^{i\psi} \underbrace{P(\psi, t)}_{\text{pdf}}$$

pdf → anticipates Fokker-Planck.

$\therefore$

$$\frac{d\psi_k}{dt} = \frac{\epsilon}{N} \sum_{j=1}^{N} \sin(\psi_j - \psi_k) + \xi_k(t)$$

$$= \epsilon\left(-X \sin\psi_k + Y \cos\psi_k\right) + \xi_k(t)$$

and can immediately write Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \psi}\left[\left\langle \frac{d\psi}{dt} \right\rangle P - \frac{\partial}{\partial \psi} \frac{\langle \delta\psi \delta\psi \rangle}{2\Delta t} P\right]$$

$$\frac{d\langle\psi\rangle}{dt} = -\underbrace{\epsilon X}_{\text{mean field}} \sin\psi + \epsilon Y \cos\psi$$

$$\frac{\langle \delta\psi \delta\psi \rangle}{2\Delta t} = \sigma^2 \quad \text{(usual white noise)}$$

$$\boxed{\frac{\partial P}{\partial t} = \epsilon \frac{\partial}{\partial \psi}\left[(x \sin\psi - y\cos\psi) P \right] + \nabla^2 \frac{\partial^2 P}{\partial \psi^2}} \quad *$$

$$\left\{ \begin{array}{l} \text{Fokker-Planck Eqn} \\ \text{for } P \end{array} \right.$$

$$\left. \begin{array}{l} x = \int_0^{2\pi} d\psi \, P(\psi, t) \cos\psi \\ y = \int_0^{2\pi} d\psi \, P(\psi, t) \sin\psi \end{array} \right\} \text{relates Mean Field to pdf}$$

N.B.: Fokker-Planck Equation now nonlinear! (a la' Landau Collision Integral).

Now, to solve (*) above:

$$P(\psi, t) = \underbrace{\frac{1}{2\pi}}_{\text{normalization.}} \sum_l P_l(t)\, e^{il\psi}$$

Can observe:

→ normalization of zero mode ⇒ $P_0 = 1$

→ $Z = x + iy$ so $Z = P_1$ $\quad (P_{-1} = P_1^*)$

Now, nonlinearity of F.P.E. couples harmonics ⇒

$$\frac{dP_\ell}{dt} = \underbrace{-\sigma^2 \ell^2 P_\ell}_{\text{diffusion}} + \underbrace{\frac{\ell\epsilon}{2}\Big(\underbrace{P_{\ell-1}P_1}_{\substack{\text{"birth"}\\ \ell-1\to\ell}} - \underbrace{P_{\ell+1}P_1^*}_{\substack{\text{"death"}\\ \ell\to\ell+1}}\Big)}_{\text{drift}}$$

(mean field ⟺ $\ell = 1$).

so ⇒

$$\dot{P}_1 = \frac{\epsilon}{2}\left(P_1 - P_2 P_1^*\right) - \sigma^2 P_1$$

$$\dot{P}_2 = \epsilon\left(P_1^2 - P_3 P_1^*\right) - 4\sigma^2 P_2$$

$$\dot{P}_3 = \frac{3\epsilon}{2}\left(P_2 P_1 - P_4 P_1^*\right) - 9\sigma^2 P_3$$

have hierarchy of coupled mode amplitude equations.

Now, can observe:

$\Rightarrow$ $\rho_0 = 1$, $\underset{\ell \neq 1}{\rho_\ell} = 0$ is solution.

unstable if: $\longrightarrow$ $(\epsilon > 2\sigma^2)$

$$\dot{\rho}_1 = \left(\frac{\epsilon}{2} - \sigma^2\right)\rho_1 - \frac{\epsilon}{2}\cancel{\rho_2}\rho_1^*$$

$\Rightarrow$ Now, near threshold:

$$\epsilon \sim 2\sigma^2$$

$$\frac{\dot{\rho}_1}{\rho_1} \cong O(\epsilon - 2\sigma^2)$$

$$\frac{\dot{\rho}_2}{\rho_2} \cong -4\sigma^2$$

$\Rightarrow$ $\ell = 2$ decays rapidly compared to $\ell = 1$

$\therefore$ can slave ρ_2 to ρ_1, i.e.:

$$\underset{\sim 0}{\cancel{\dot{\rho}_2}} = \epsilon\left(\rho_1^2 - \rho_3\rho_1^*\right) - 4\sigma^2\rho_2$$

$$\Rightarrow \rho_2 = \frac{\epsilon}{4\sigma^2}\rho_1^2$$

and plugging into ρ_1 eqn:

$$\dot{\rho}_1 = \left(\frac{\epsilon}{2} - \sigma^2\right)\rho_1 - \frac{\epsilon}{2}\left(\frac{\epsilon}{4\sigma^2}\right)\rho_1^2\rho_1^*$$

$$\Rightarrow \dot{\rho}_1 = \left(\frac{\epsilon}{2} - \sigma^2\right)\rho_1 - \frac{\epsilon^2}{8\sigma^2}|\rho_1|^2\rho_1$$

Yet again, Landau-Stuart!

$\therefore$ can calculate $|Z|^2$ near threshold:

$$|Z|^2 = (\epsilon - 2\sigma^2)\frac{4\sigma^2}{\epsilon^2}$$

Note: $\rightarrow$ $$|Z|^2 = \left(\frac{\epsilon - \epsilon_{crit}}{\epsilon}\right)\frac{4\sigma^2}{\epsilon}$$

$$\alpha = 1/2$$

$\rightarrow$ noise level sets threshold.

N.B. $\Rightarrow$ $\Bigg\{$ Interesting question? - How much "noise" does it take to "randomize" phase i.e. make pdf $\{\phi\}$ $\rightarrow$ Gaussian?
(width coefficient)

- what is effect of quenched disorder in couplings on phase pdf?

C.) Variations and Extensions

Have two basic models: $\sin(\phi_i - \phi_j)$ coupling

i.) Uniform coupling Kuramoto transition with $g(\omega)$

ii.) Uniform coupling Kuramoto transition with $\omega = \omega_0$ and noise.

Can immediately state several variations on the basic themes of i.) and ii.):

a.) combine i.) and ii.) $\Rightarrow$ distribution of frequencies <u>and</u> noise

b.) generalized attractive coupling, i.e.

$$\epsilon\left(\frac{1}{N}\right)\sum_{j=1}^{N}\sin(\phi_j - \phi_k) \rightarrow \frac{\epsilon}{N}\sum_{j=1}^{N} \mathcal{Z}(\phi_j - \phi_k)$$

i.e. order parameter $\ddot{?}$ transition behavior

c.) random coupling $\rightarrow$ "oscillator spin glass"

<u>i.e.</u>

$$\frac{d}{dt}\phi_k = \omega_k + \frac{1}{2\pi}\sum_{i=1}^{N}\tilde{J}_{ij}\sin(\phi_i - \phi_k + \alpha_{ij})$$

where $\begin{cases} \tilde{J}_{ij} \geq 0 & \text{is random} \\ \alpha_{ij} & \text{random or fixed (to produce frustration)} \end{cases}$

d.) hysteretic transition → add inertia to the phase, i.e.

$$m \frac{d^2\phi_k}{dt^2} + \frac{d\phi_k}{dt} = \omega_k + \frac{\epsilon}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_k) + \xi_k(t)$$

Some comments on each:

a.) adding noise to K-transition:

$$\frac{d\phi_k}{dt} = \omega_k + \xi_k(t) + \frac{\epsilon}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_k)$$

so need F.P.E. for $P(\phi, \omega, t)$.

For mean field limit $(N \to \infty)$:

$$K e^{i\Theta} = \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} d\omega \, e^{i\phi} \, P(\phi, \omega, t) \, \underbrace{g(\omega)}_{\text{given distribution}}$$

⇒ Problem #1:

Develop the theory of the K-transition with noise.

Hint: → Derive a F.P.E. for the pdf $\rho(\phi, \omega, t)$ with mean-field coupling
→ approach via truncation of harmonic coupling
→ Consider sensitivity to $g(\omega)$ → what type of bifurcation occurs.
→ get as far as possible. What would a simple minded experimentalist observe for this system?

refs: Bonilla, et. al. '92; Acebron et. al. '98.

b.) for generalized attractive coupling

$$\frac{d\phi_k}{dt} = \omega_k + \frac{\epsilon}{N}\sum_{j=1}^{N} z(\phi_j - \phi_k)$$

in general: $z(\phi) = \sum_{\ell} z_\ell e^{2\pi i \ell \phi}$

so, for oscillators mean field, can generalize order parameter:

$$Z_\ell = \frac{1}{N} \sum_{k=1}^{N} e^{i\ell(\phi_k - \bar{\omega} t)}$$

ℓ^{th} harmonic

$$= \frac{1}{N} \sum_{k=1}^{N} e^{i\ell\psi_k}$$

so

$$\frac{d\phi_k}{dt} = \omega_k - \epsilon H(\phi_k - \bar{\omega} t)$$

$$H(\psi) = -\sum_\ell q_\ell Z_\ell e^{-i\ell\psi}$$

order function $\Rightarrow$ mean "forcing" on each oscillator phase.

Generalizes K- model

Obviously, $H \neq 0 \Rightarrow$ synchronization.

$\Rightarrow$ Problem #2:

a) Work out the theory of the K- transition for generalized harmonic q.

b.) show, near the synchronization threshold:

$$\| H \| \sim \epsilon - \epsilon_{c}$$ in contrast to

the result for the K-model $\left(K \sim (\epsilon - \epsilon_c)^{1/2} \right)$

c.) What happens when external noise is added? Show:

$$|\rho_e| \sim \left[(\epsilon - \epsilon_{crit})(\epsilon - \epsilon_{crit} + \ell^2 \sigma^{-2}) \right]^{1/2}$$

refs: H. Daido: '92, '93, '95.
A comprehensive reference is:

H. Daido, Physica D 91 24-66 (1996).

For noise; see J.D. Crawford, '96.

Get as far as you can.

c.) With random coupling,

$$\frac{d\phi_k}{dt} = \omega_k + \sum_{i=1}^{N} \tilde{J}_{ik} \sin(\phi_i - \phi_k + \alpha_{ik})$$

Here: $\Rightarrow \omega_i$ distributed according to $g(\omega)$
(but could take constant and
add noise, also)

$$\Rightarrow \begin{cases} \text{Pdf}(J_{ij}) = (2\pi J^2/N)^{-1/2} \exp(-N J_{ij}^2/2J^2) \\ J_{ij} \text{ random, but fixed} \end{cases}$$

Problem#3 (Challenging):

i.) Review the Sherrington-Kirkpatrick model of spinglasses. Summarize its salient points and results, and discuss the correspondence with this problem.

ii.) Take $\alpha_{ij} = 0$. Develop the theory of clustering for the random coupling model:

A.) Try first <u>without</u> consulting the literature on spin glasses (c.f. Binder).

B.) Now, consult the literature and repeat. You may find the Glauber model/theory useful

C.) Summarize numerical findings, and compare with your results.

<u>c.f.</u> H. Daido, Phys. Rev. Lett. <u>68</u> 1073 (1992).

(iii) Take $\alpha_{ij} \neq 0$ and either 0 or π (attractive / repulsive coupling), with equal probability. Discuss the effects of "frustration".

Hint: "Consider each oscillator in a 'local field'"

$$\rho_K = \frac{1}{J} \sum_{j=1}^{N} J_{kj} e^{i\phi_j} .$$

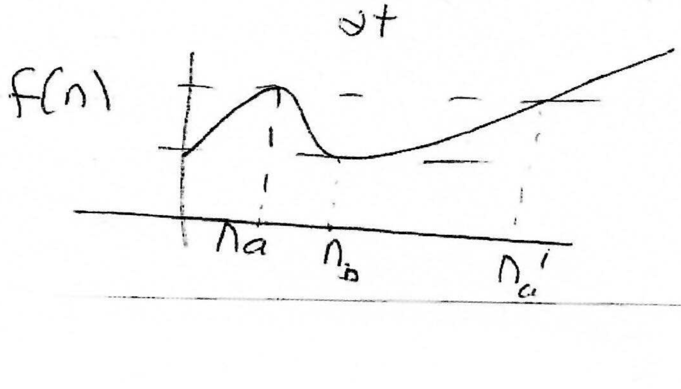
What happens as J increases?

See also: Bonilla, et. al. '93; Park, et. al. '98.

d.) hysteretic transition

With "phase inertia", transition from asynchronous → synchronous states exhibits hysteresis.

i.e. recall: $\frac{\partial n}{\partial t} = f(n) - S$



$f(n)$ bistable

two stable branches : $n < n_a$ – (I) root
$n > n_b$ – (II) root

forward transition : (I) root disappears at n_a
(I) → (II) ⇒ transition to (II) root

back transition: (II) root persists to n_b
does not disappear at n_a.

→ hysteresis

Problem #4 : Show that hysteresis is demonstrated in the transition between asynchronous and synchronous states.

Cf: Tanaka, et. al. '97 ; Hong, et. al. '99.

Next : – Phase Dynamics in Space and Time
– Spiral Waves in reaction-diffusion systems.