# Convection Patterns

Physics 221A, Spring 2017
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## 1 Introduction

In previous lectures, we have studied the basics of dynamics, which include dimensions of (strange) attractors, Lyapunov exponents, and chaos. Then, we have studied patterns in time, i.e. phase dynamics described by the Complex Ginzburg Landau (CGL) equation. We started with the single oscillator synchronization with noise. As local coupling bing considered, phase diffusion (KPZ) model was introduced, which gives domains of synchronization and the probability distribution of phase. In the case with global coupling, the coupling competes against dispersion and noise, and can result in global synchronization (Kuramoto transition), where range of couping is the key. Finally, we studied phase turbulence, i.e. repulsive coupling and Kuramoto-Sivashinsky equation.

In this chapter, we study patterns in space, i.e. patterns formed by convections near marginality. The focus is on secondary instability in an ensemble of convection cells/rolls near marginality, i.e. in the weakly nonlinear regime. Fig.1 shows the two major types of secondary instabilities discussed here. Eckhaus instability arises from modulating the array of convection rolls, which breaks the translational symmetry, resulting in clustering and coarsening of vortices. Zigzag instability is due to the bending of the rolls, which breaks the rotational symmetry.

The subject discussed here is classic, as easily amenable to experiment, and is tractable as "near equilibrium". The approach is to set up a basic model of near marginal convection rolls, i.e. the Swift-Hohenberg model. Here, we adopt the envelope formalism for patterns. Modulations in ensemble of rolls, i.e. pattern dynamics on scales larger than that of individual rolls, are described by envelope equations (Newell-Whitehead model), which is similar to the wave kinetic equation (WKE). In the Newell-Whitehead model, the translational and rotational invariance is not preserved. Thus, secondary instabilities arise due to the asymmetries, leading to pattern formation. We also show that the nonlinear Schrodinger equation is a generic envelope equation of a weakly nonlinear dissipative wave train.

The subject discussed here is more elegant and "classical" than useful, and is a *must* for a basic course. Some related, but more relevant models include 1) fixed flux convection (Chapman-Proctor model) and 2) convection driven flows (Howard-Krishnamurti model).

## 2 Physics of Convection

### 2.1 Rayleigh-Benard Convection

We consider the Rayleigh-Benard convection in stratified fluid or atmosphere. The fluid is incompressible, i.e. $\nabla \cdot \mathbf{v} = 0$, as $\tau > l/c_s$. A parcel will rise by buoyancy if $dS/dz < 0$, i.e.

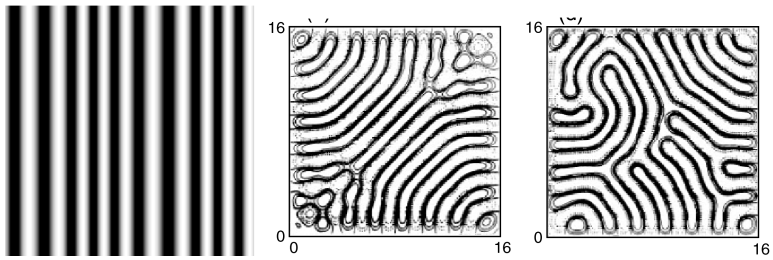


Figure 1: Patterns formed by 2 types of secondary instabilities in near marginal convections: (left) Eckhaus instability, (center) zigzag instability, and (right) coexistence of both instabilities.

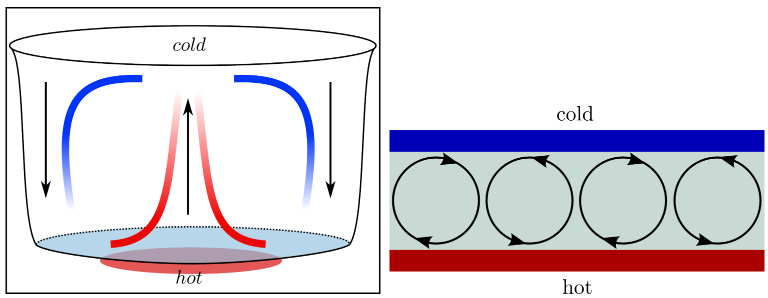


Figure 2: Cartoon of (left) Rayleigh-Benard convection and (right) convection rolls.

$$\frac{1}{T}\frac{dT}{dz} < \frac{\gamma - 1}{\rho}\frac{d\rho}{dz}, \tag{1}$$

where $\gamma$ is the ratio of heat capacities. The characteristic time scale of the buoyancy ($\tau_b$) is

$$\frac{1}{\tau_b^2} \cong \frac{g}{\gamma}\frac{\partial S_0}{\partial z}. \tag{2}$$

If there exists dissipation, then the time derivatives become

$$\partial_t \tilde{T} \to \partial_t \tilde{T} - \chi \nabla^2 \tilde{T},$$

$$\partial_t \tilde{\mathbf{v}} \to \partial_t \tilde{\mathbf{v}} - \nu \nabla^2 \tilde{\mathbf{v}}.$$

The viscosity and heat diffusion can damp convection. Therefore, it is natural to require for instability that

$$\frac{\tau_\nu \tau_\chi}{\tau_b} > 1. \tag{3}$$

Hence, we can conveniently define the Rayleigh number as

$$Ra \equiv g\frac{\partial S}{\partial z}\frac{l^4}{\nu\chi}, \tag{4}$$

such that instability requires $Ra > Ra_{crit}$. In a 2D box with hight $h$, the Rayleigh number is

$$Ra \equiv \frac{g\Delta T \alpha h^3}{\nu\chi}, \tag{5}$$

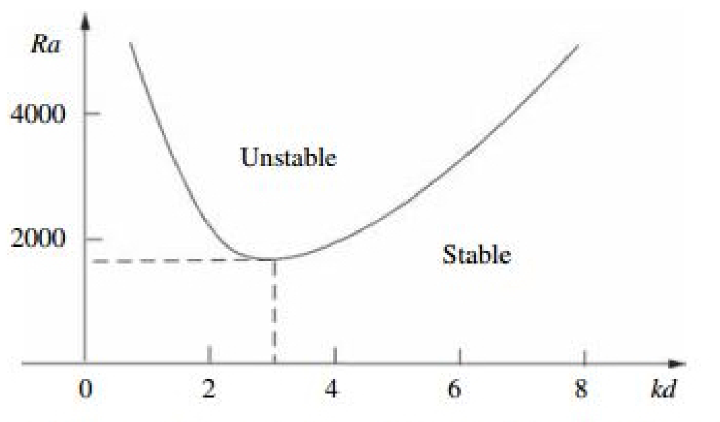


Figure 3: Critical Rayleigh number curve. $k_h h$ is the normalized horizontal wave number. Minimum $Ra_{crit}$ occurs at $k_{h,crit}$.

where $\alpha$ is the coefficient of thermal expansion, i.e. $\delta\rho = -\alpha\delta T$.

The basic equations of Rayleigh-Benard convection are

$$\frac{\partial}{\partial t}\nabla^2\phi - \nu\nabla^2\nabla^2\phi = -g\alpha\frac{\partial}{\partial x}\left(\frac{\tilde{T}}{T_0}\right), \tag{6}$$

$$\frac{\partial}{\partial t}\left(\gamma\frac{\tilde{T}}{T_0}\right) - \chi\nabla^2\tilde{T} = -\tilde{v}_z\frac{\partial S_0}{\partial z}. \tag{7}$$

Here, the flow velocity is defined as $\mathbf{v} = -\nabla\phi \times \hat{y}$, and $\nabla^2 = \partial_x^2 + \partial_z^2$. Redefining $w = \tilde{v}_z$, $\theta = \tilde{T}/T_0$, and $\beta = -\partial S_0/\partial z$, we can rewrite Eq.(6)-(7) as

$$\frac{\partial}{\partial t}\nabla^2 w = g\alpha\frac{\partial^2\theta}{\partial x^2} + \nu\nabla^2\nabla^2 w, \tag{8}$$

$$\frac{\partial\theta}{\partial t} = \beta w + \chi\nabla^2\theta. \tag{9}$$

The behavior of the critical Rayleigh number is set by dissipation and boundary conditions. For no slip boundary conditions: $\tilde{v}_z|_{0,h} = 0$ and $\tilde{v}_h|_{0,h} = 0$, where $\tilde{v}_h$ is the horizontal velocity. Note that for incompressible flows, $\nabla_h\tilde{v}_h + \partial_z\tilde{v}_z = 0$, and $\nabla_h\tilde{v}_h|_{0,h} = 0$ due to the no slip boundary conditions. Hence, $\partial_z\tilde{v}_z|_{0,h} = 0$. The critical Rayleigh number depends on the wave number in the horizontal direction, as shown by Fig.3. For high $k_h$, the dissipation ($\nu k_h^2$ etc.) stabilizes convection. For low $k_h$, convection is damped by the no slip boundary, where $\tilde{v}_h = 0$. Note that in the stress free regime, the critical $Ra$ curve has the similar structure, while numbers change.

## 2.2 Swift-Hohenberg Model

How to describe the convection for $Ra = Ra_{crit} + \epsilon$, i.e. a small excursion into super critical state? In 1D, the key elements are $Ra_{crit}$, $k_{crit}$, the $Ra_{crit}$ curve, and the saturation. Because the system

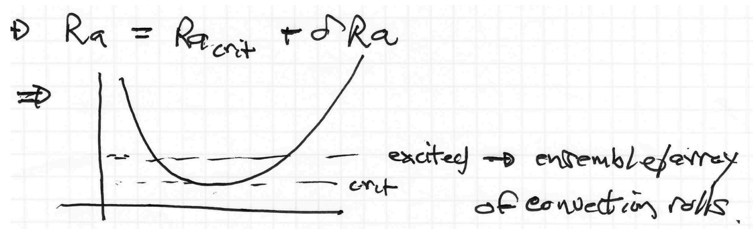


Figure 4: Swift-Hohenberg model is applicable to a band of modes near $Ra_{crit}$.

approaches the steady state, so the $Ra_{crit}$ curve can be fit with a parabola, and thus the model of growth near marginality is

$$\gamma\tau_0 = (Ra - Ra_{crit}) - \tau_0\varepsilon_0^2(k - k_{crit})^2. \tag{10}$$

Therefore, schematically, we can obtain

$$\tau_0\frac{\partial w}{\partial t} = (Ra - Ra_{crit})w - \tau_0\varepsilon_0^2\left(\sqrt{-\partial_x^2} - k_{crit}\right)^2 w - |w|^2 w, \tag{11}$$

which has the form of the Landau equation. This is a step toward the Swift-Hohenberg Model, a reduced model of convection near marginality [Swift, Hohenberg (1977)].

Beyond 1D, we consider uniform base state and rotationally invariant in 2D plane (i.e. convection rolls). Thus, $\gamma_{\mathbf{q}}$ can depend only on $|\mathbf{q}| = q$, not $\mathbf{q}$. Hence, we can obtain

$$\gamma_q\tau_0 = p - c(q - q_c)^2, \tag{12}$$

$$\tau_0\frac{\partial w}{\partial t} = \left[p - c\left(\sqrt{-\nabla^2} - q_c\right)^2\right] w - |w|^2 w, \tag{13}$$

Here, $p \equiv Ra - Ra_{crit}$ is the control parameter, and $q_c$ is the wave number with the minimum growth rate.

Near the onset, there is $q + q_c \cong 2q_c$, and so we can write the number 1 in a "creative way": $1 \cong (q + q_c)^2/4q_c^2$. Consequently, after rescaling, the growth rate is rewritten as

$$\gamma_q\tau_0 \cong p - \frac{c}{4q_c^2}(q^2 - q_c^2)^2. \tag{14}$$

Finally, we can obtain the Swift-Hohenberg equation

$$\frac{\partial w}{\partial t} = rw - c\left(\nabla^2 + q_c^2\right)^2 w - |w|^2 w. \tag{15}$$

The nonlinear term restricts the growth at finite amplitude, and respects the inversion symmetry ($w \to -w$) and phase symmetry of the basic equations.

The Swift-Hohenberg model describes convection near the onset, but cannot quantify its own breakdown. It describes interactions above marginality, with a narrow band of modes (Fig.4).

The Swift-Hohenberg model can be derived from basic equations systematically, but the derivation is tedious and not instructive. Alternatively, the Swift-Hohenberg model is derivable from the variational principle, i.e. by considering the Lyapunov function:

$$V[w] = \int dx \int dy \left\{ -\frac{1}{2} r w^2 + \frac{1}{4} r w^4 + \frac{1}{2} \left[ \left( \nabla^2 + q_c^2 \right) w \right]^2 \right\}, \tag{16}$$

and

$$\frac{dV}{dt} = - \int dx \int dy \, (\partial_t w)^2 . \tag{17}$$

This means any evolution of $w$ tend to decrease $V$, and $V$ is minimal at the stationarity of $w$. It can easily be shown that $\partial_t w = -\delta V / \delta w$, from which the Swift-Hohenberg model can be obtained.

# 3 Pattern Formation: Secondary Instability

How does the pattern of excited cells evolve? What configuration does it adopt? In this section, we first derive the Newell-Whitehead model, which describes the stability of modulation to the base state. There are 2 types of secondary instabilities arising from this system, corresponding to the 2 invariance in the base state. The Eckhaus instability emerges from breaking the translational symmetry, resulting in the coarsening of convection domains. The zigzag instability arises from bending the convection cells, with the rotational symmetry broken.

## 3.1 Newell-Whitehead Model

It is classic to explore the stability of a band of modes with

$$\mathbf{q} = q_0 \hat{x} + \mathbf{k}. \tag{18}$$

The base state is an array of convection cell with wave number $q_0 \hat{x}$. $\mathbf{k}$ represents a large-scale 2D perturbation to the base state, which breaks the symmetry. Because $|\mathbf{k}|/|q_0| \ll 1$, so $w$ can be written in terms of an amplitude and a carrier, i.e.

$$w \cong w_0 A(x, y, t) e^{i q_0 x} + c.c. + O(\epsilon). \tag{19}$$

Now, go back to the model of growth near marginality, i.e. $Ra - Ra_{crit} = \epsilon \ll 1$, and we can obtain

$$\begin{aligned} \gamma \tau_0 &= \epsilon - \varepsilon_0^2 \left( |q_0 \hat{x} + \mathbf{k}| - q_0 \right)^2 \\ &= \epsilon - \varepsilon_0^2 \left\{ q_0 \left[ \left( 1 + \frac{k_x}{q_0} \right)^2 + \frac{k_y^2}{q_0^2} \right]^{1/2} - q_0 \right\}^2 \\ &\cong \epsilon - \varepsilon_0^2 \left( k_x + \frac{k_y^2}{2 q_0} \right)^2 . \end{aligned}$$

The second term here describes the envelope's dependence on $\mathbf{k}$. Higher order terms in $\mathbf{k}$ are irrelevant. Note that the asymmetry in $k_x$ and $k_y$ is due to symmetry breaking, or direction set, of the base state.

Using the same technique, we can obtain the envelope equation by considering

$$\tau_0 \partial_t w = \epsilon w - \varepsilon_0^2 \left(|q_0 \hat{x} + \mathbf{k}| - q_0\right)^2 w. \tag{20}$$

Here, we can set $w = w_0 A \exp(iq_0 x)$, and $\mathbf{k} \to -i\partial_{\mathbf{x}}$. Moreover, considering the $w \to -w$ symmetry and $w \to we^{i\phi}$ symmetry, the nonlinear saturation is retained as $-|w|^2 w$. Finally, we obtain the Newell-Whitehead equation (amplitude equation)

$$\partial_T A = A + \left(\partial_x - \frac{i}{2}\partial_y^2\right)^2 A - |A|^2 A, \tag{21}$$

where we have rescaled the following quantities: $x = |\epsilon|^{1/2} x/\varepsilon_0$, $y = |\epsilon|^{1/4} y (q_0/\varepsilon_0)^{1/2}$, $T = \epsilon t/\tau_0$, $A = (g/|\epsilon|)^{1/2}$.

Similar to the Swift-Hohenberg model, the Newell-Whitehead model has Lyapunov function, and so is also derivable from the variational principle. The key difference between the two models is that the Swift-Hohenberg model maintains the rotational symmetry of the base state, while the Newell-Whitehead model breaks the symmetry by considering an asymmetric 2D perturbation to the base state, i.e. $\mathbf{q} = q_0 \hat{x} + \mathbf{k}$.

The Newell-Whitehead equation Eq.(26) is a complex equation, which guarantees that phase dynamics is relevant. Thus, it is useful to rewrite as $A = |A|e^{i\phi} \equiv ae^{i\phi}$. Ignoring the $y$ dependence, we can obtain an amplitude equation and a phase equation:

$$\partial_T a = \left[1 - (\partial_x \phi)^2\right] a + \partial_x^2 a - a^3, \tag{22}$$

$$\partial_T \phi = \partial_x^2 \phi + \frac{2\partial_x a}{a} \partial_x \phi. \tag{23}$$

The phase evolution is driven by phase diffusion and an extra term $2\partial_x \phi \partial_x a/a$. Hence, the sign of the effective diffusivity depends on the magnitude and sign of this extra term. We can observe that the amplitude $a$ evolves via $a - a^3$, favoring long wavelengths. Therefore, the short wavelength dependent terms in the amplitude equation are of the same order, giving that $(\partial_x \phi)^2 \sim \partial_x^2 a/a \sim (\partial_x a)^2 / a^2$. Therefore, the phase evolution is of order $\partial_T \phi \sim \partial_x^2 \phi \pm 2 (\partial_x \phi)^2 \sim -k_x^2 \phi (1 \pm 2\phi)$. Here, the sign is determined by the local slope of the amplitude. When $|\phi| > 1/2$, the second term on the left hand side of the phase equation can effectively flip the sign of effective diffusivity.

Next, we calculate the steady state phase winding solution to the Newell-Whitehead equation. For simplicity, consider $\partial_x a = 0$ and $\phi = \delta k x + \phi_0$. From the amplitude equation, we can obtain

$$a = \sqrt{1 - \delta k^2}, \tag{24}$$

which requires the rescaled $\delta k < 1$, or the original $\delta k < \sqrt{\epsilon}/\varepsilon_0$. This suggests that phase winding is termed with long wavelength. Hence, the solution is

$$w = \frac{1}{2}\left(\sqrt{1 - \delta k^2} e^{i\phi_0} e^{iq_0 x} e^{i\delta k x} + c.c.\right). \tag{25}$$

Here, the “winding” is a weak modulation of the mode amplitude.

### 3.2 Eckhaus Instability

What type of secondary instabilities might occur? To answer this question, we need to look at what symmetry of the base state is broken. The base state has translational symmetry and rotational symmetry. Next, we examine the secondary instabilities induced by breaking these two symmetries

Breaking the translational symmetry results in the Eckhaus instability. Starting from uniformly distributed convection rolls, modulation brings rolls closer. Then, vortices with like signs attract and collapse to roll pairs, which lowers the energy $(\nabla A)^2$. The final result is patterns of vortex bunches. This process is one of the modulation → coalescence → condensation phenomena. The clustering of vortices suggests the existence of negative diffusion.

With a modulation of the amplitude $A(x,y,t) = \tilde{A}(x,y,t)e^{i\delta kx}$, the Newell-Whitehead equation becomes

$$\partial_T \tilde{A} = \left(1-\delta k^2\right)\tilde{A} + 2i\delta k\left(\partial_x - i\partial_y^2\right)\tilde{A} + \left(\partial_x - \partial_y^2\right)^2 A - |\tilde{A}|^2\tilde{A}. \tag{26}$$

The uniform phase winding solution is $\tilde{A}_0 = \sqrt{1-\delta k^2}$, requiring the rescaled $\delta k < 1$, or the original $\delta k < \sqrt{\epsilon}/\varepsilon_0$.

Consider a perturbation about the uniform phase winding state, i.e. $\tilde{A} = \tilde{A}_0 + \tilde{a}$. The perturbed amplitude is complex, i.e. $a = u + iv$. Therefore, equations for $u$ and $v$ can be obtained

$$\partial_t u = \left[-2(1-\delta k^2) + \partial_x^2 + \delta k\partial_y^2 - \partial_y^4\right]u - (2\delta k - \partial_y^2)\partial_x v, \tag{27}$$

$$\partial_t v = (2\delta k - \partial_y^2)\partial_x u + (\partial_x^2 \delta k \partial_y^2 - \partial_y^4)v. \tag{28}$$

Writing $u$ and $v$ in Fourier components, i.e. $u = ue^{st}\cos(q_x x)\cos(q_y y)$, $v = ve^{st}\cos(q_x x)\cos(q_y y)$, the dispersion relation can be obtained

$$s^2+2(1-\delta k^2+q_x^2+q_y^2\delta k+q_y^4)s+[2(1-\delta k^2)+q_x^2+q_y^2\delta k+q_y^4](q_x^2+q_y^2\delta k+q_y^4)-q_x^2(2\delta k+q_y^2)^2 = 0. \tag{29}$$

For Eckhaus instability, it is uniform in the $\hat{y}$ direction, i.e. $q_y = 0$. Hence, the dispersion relation becomes

$$s^2 + 2(1-\delta k^2 + q_x^2)s + q_x^2[2(1-3\delta k^2) + q_x^2] = 0. \tag{30}$$

which can be rewritten in the generic form

$$(s-s_1)(s-s_2) = 0, \text{ or } s - (s_1+s_2)s + s_1 s_2 = 0. \tag{31}$$

Instability, i.e. complex solution, requires $s_1 s_2 < 0$. Therefore, the criterion for onset of Eckhaus instability is

$$\delta k > \frac{\sqrt{\epsilon/3}}{\varepsilon_0}. \tag{32}$$

Combining the requirement for phase winding, the Eckhaus instability requires $\sqrt{\epsilon/3} < |\delta k| < \sqrt{\epsilon}$, for $\varepsilon_0 = 1$. This hydrodynamic mode results from the broken translational invariance. It can also be presented as a negative diffusion phenomenon.

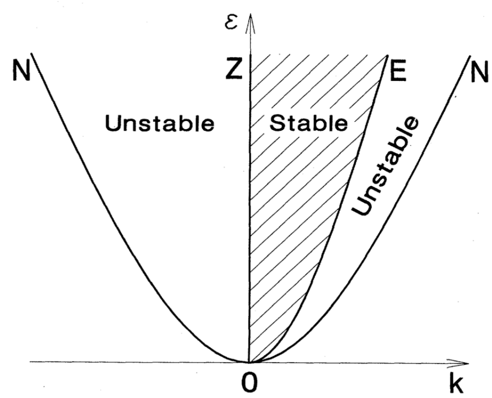


Figure 5: Criteria of Eckhaus instability and zigzag instability. Here, $\varepsilon \equiv Ra - Ra_{crit}$, $k \equiv \delta k$, N is the neutral curve below which is the stable region, E is the Eckhaus unstable region, and Z is the zigzag unstable region.

### 3.3 Zigzag Instability

Another invariance is the rotational invariance in the $x, y$ plane of the primary instability. It is broken by bending the convection rolls. The bending saturates due to the $(\partial_y^2 A)^2$ term in the energy of Newell-Whitehead model.

In analysis, take $q_x = 0$, $q_y$ finite, and then we can obtain the solutions of Eq.(29)

$$s_- = -2(1 - \delta k^2) - q_y^2 \delta k - q_y^4, \tag{33}$$

$$s_+ = -q_y^2 (q_y^2 + \delta k). \tag{34}$$

$s_-$ is dominated by the first term, so is negative. Unstable solution requires $s_- s_+ < 0$. Thus, $s_+ > 0$, and we can obtain the criterion for zigzag instability

$$\delta < -q_y^2. \tag{35}$$

## 4 Other Pattern-Related Problems

Below is a list of problems relevant to the topic of pattern formation discussed here:

- Secondary instability in waves: nonlinear Schrodinger (NLS) equation, which is a special case of the complex Ginzburg Landau equation.
- Geometry effect: effects of boundary conditions on amplitude equation. Obviously, the wave number $q$ is quantized within the boundary layer. (Fig.7)
- Grain boundary/dislocations, resulting in pattern discontinuities. (Fig.8)
- Phase diffusion: negative diffusion during the clustering process.

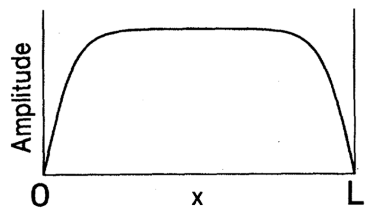


Figure 6: Effects of boundary layer on amplitude.

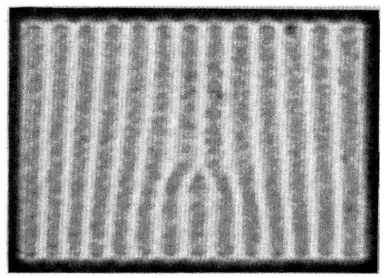

Figure 7: Effects of grain boundary/dislocation.

### 4.1 Nonlinear Schrodinger Equation as Envelope Equation

The classic plasma problem of Langmuir turbulence reduces to NLS in the subsonic case. But NLS is *far* more *generic*. Also, we observe that NLS equation can be derived from the CGL equation

$$\partial_t A = rA + (1 + i\alpha)\partial_x^2 A - (1 + i\gamma)|A|^2 A, \tag{36}$$

where $\alpha$ is the dispersion effect, and $\gamma$ is the nonlinear frequency shift. Taking the conservative limit $r \to 0$, $|\alpha|, |\gamma| \gg 1$, we can obtain

$$i\partial_t A = -\alpha\partial_x^2 A + \gamma|A|^2 A, \tag{37}$$

which is the nonlinear Schrodinger equation. Therefore, NLS is generic, and it gives quadratic feedback on the mean field, through the $|A|^2 A$ term.

Next, we show the NLS can describe the envelope of a weakly nonlinear dissipative wave train. Consider a linear, dispersive wave train

$$\Phi = \int dk F(k) \exp(ikx - i\omega(k)t). \tag{38}$$

With a small perturbation to the wave number, i.e. $k_0 \to k_0 + \delta k$, the dispersion relation becomes

$$\begin{aligned} \omega &= \omega(k_0 + k - k_0) \\ &= \omega_0 + (k - k_0)\omega_0' + \frac{(k - k_0)^2}{2}\omega_0'' + O(\delta k^3). \end{aligned}$$

Ignoring the higher order terms, the wave train can be written as

$$\Phi = \varphi e^{i(k_0 x - \omega_0 t)}, \tag{39}$$

where the envelope function is

$$\varphi = \int dk F(k+k_0) \exp\left[ikx - i\left(k\omega_0' + \frac{k^2}{2}\omega_0''\right)t\right]. \tag{40}$$

The envelope $\varphi$ clearly satisfies

$$i(\partial_t \varphi + \omega_0' \partial_x \varphi) + \frac{1}{2}\omega_0'' \partial_x^2 \varphi = 0. \tag{41}$$

Hence, the modulation has the dispersion relation

$$\omega = k\omega_0' + \frac{1}{2}k^2\omega_0'', \tag{42}$$

i.e. $\varphi = a_0 \exp(ikx - i\omega t)$. By adding a nonlinear frequency, we can obtain the new dispersion relation

$$\omega = k\omega_0' + \frac{1}{2}k^2\omega_0'' - qa^2. \tag{43}$$

Then, the envelope satisfies

$$i(\partial_t \varphi + \omega_0' \partial_x \varphi) + \frac{1}{2}\omega_0'' \partial_x^2 \varphi + q|\varphi|^2\varphi = 0. \tag{44}$$

Now, in the frame co-moving with the group velocity of the wave train, i.e. $v_{gr} \equiv \omega_0'$, the envelope equation becomes

$$i\partial_t \varphi + \frac{1}{2}\omega_0'' \partial_x^2 \varphi + q|\varphi|^2\varphi = 0, \tag{45}$$

which is the nonlinear Schrodinger equation. Therefore, NLS is generic to weakly nonlinear dissipative wave train, with nonlinear frequency shift $\sim a^2 \sim |\varphi|^2$.

NLS has imaginary coefficients, so we need to treat it via $\varphi = Ae^{i\phi}$. The amplitude can be further separated into a spatially uniform mean field and a small fluctuation, i.e. $A = A_0(1 + \tilde{A})$. Thus the evolution of $\tilde{A}$ and $\phi$ is

$$\partial_t \tilde{A} + \frac{1}{2}\omega_0'' \partial_x^2 \phi = 0, \tag{46}$$

$$\partial_t \phi - \frac{1}{2}\omega_0'' \partial_x^2 \tilde{A} - \frac{1}{2}\omega_0'' \left(\partial_x \phi\right)^2 + q|A_0|^2 \tilde{A} = 0. \tag{47}$$

By linearizing these two equations, we can obtain the dispersion relation

$$\Omega^2 = \frac{1}{4}\omega_0''^2 k^4 + \frac{1}{4} q\omega_0'' k^2 |A_0|^2. \tag{48}$$

Therefore, instability requires $q\omega_0'' < 0$. This is the Benjamin-Feir instability, where the modulation grows. It is also the linear relative of self-focusing.

The self-focusing effect originates from the nonlinear attraction ($|\varphi|^2\varphi$), while the diffraction ($\nabla^2\varphi$) acts against focusing. The competition of the two depends on the dimensionality of the system. When the wave field intensity $|\varphi|^2$ is localized in a $n$D region whose volume is $\sim l^n$. The total energy of the system $\sim |\varphi|_l^2 l^n$ is conserved, while the region contracts or expands. Therefore, when the region size becomes $l'$, the intensity becomes $|\varphi|_{l'}^2 \sim |\varphi|_l^2 (l/l')^n$. On the other hand, the diffraction scales as $\sim (l/l')^2$. Therefore, in 1D, there exists a scale at which the diffraction balances the self-attraction, and thus self-focusing stops, producing a soliton structure. In 3D, the self-attraction is always stronger than the diffraction, which leads to collapse of the contraction size and singularity formation. 2D is a special case, where self-attraction is always of the same order as diffraction. Thus, the system state is sensitive to perturbations and initial conditions.