Deriving the CGL (II) : $\begin{cases}\text{Reductive Perturbation} \\ \text{Theory} \Rightarrow \text{glorified form of} \\ \text{Poincaré-Lindstedt P.T.}\end{cases}$

For most cases, will be concerned with some system of oscillators, i.e.

$$\frac{dX_i}{dt} = F_i(X_1, X_2, \ldots X_n, \mu)$$

(1) $\frac{d\underline{X}}{dt} = \underline{F}(X, \mu)$

- autonomous
- $\mu > \mu_c \Rightarrow$ bifurcation from fixed/stationary state to oscillation (i.e. cycle). - Hopf bifurcation
- may possibly add diffusive coupling

now, $\underline{X}_0(\mu) \equiv$ steady solution

so, perturbing (1):

$$\underline{u} = \underline{X} - \underline{X}_0$$

and linear response matrix

$$\frac{d\underline{u}}{dt} = L\underline{u} + M\underline{u}\,\underline{u} + N\underline{u}\,\underline{u}\,\underline{u}$$

$\begin{pmatrix} L & \\ M & \text{tensors} \\ N & \end{pmatrix}$

where:

$$L_{ij} = \partial \underline{F}_i(\underline{X}_0) / \partial \underline{X}_{0j}$$

(Jacobian matrix element)

$$(M\underline{u}\underline{u})_i \equiv \sum_{j,k} \frac{1}{2!} \frac{\partial^2 F_i(\underline{x}_0)}{\partial x_{0j}\,\partial x_{0k}} u_j u_k$$

$$(N\underline{u}\underline{u}\underline{u})_i \equiv \sum_{j,k,l} \frac{1}{3!} \frac{\partial^3 F_i(\underline{x}_0)}{\partial x_{0j}\,\partial x_{0k}\,\partial x_{0l}} u_j u_k u_l$$

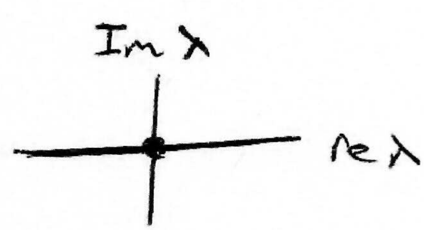
Note:

- $M\underline{u}\underline{v}$, $N\underline{u}\underline{v}\underline{w}$ symmetric in $\underline{u}, \underline{v}, \underline{w}$
- $M, N = M, N\{\mu\}$

Now, no loss of generality to take $\mu_c = 0$, so "criticality"

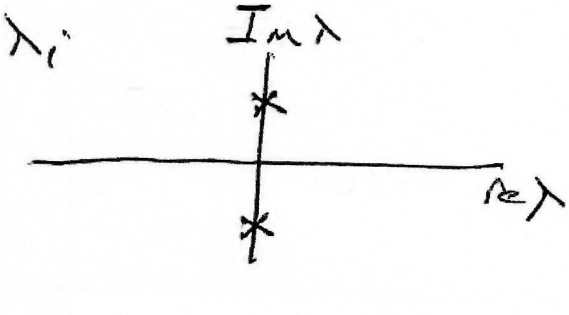
- $\mu < 0 \Rightarrow \underline{x}_0$ stable
- at least one eigenvalue $\lambda(\mu)$ $(x \sim e^{\lambda t})$ crosses $\mathrm{re}\,\lambda = 0$ as $\mu \geq 0$. Crossing can be of form:

→ $\mathrm{re}\,\lambda = 0$, $\mathrm{Im}\,\lambda = 0$



or

→ $\mathrm{re}\,\lambda = 0$, $\mathrm{Im}\,\lambda = \pm\lambda_i$



i.e. complex conjugate pairs.

and $d\,\mathrm{re}\,\lambda(\mu)/d\mu\big|_{\mu=0} > 0$

Now near criticality:

$$\underline{\underline{L}} = L_0 + \mu L_1 + \mu^2 L_2 + \dots \qquad \text{(operator)}$$

$$\begin{cases} \lambda(\mu) \equiv \text{eigenvalue going critical (i.e. re } \lambda \text{ passing zero)} \\ \bar{\lambda}(\mu) = \text{c.c.} \end{cases}$$

so
$$\begin{cases} \lambda = \lambda_0 + \mu \lambda_1 + \mu^2 \lambda_2 \\ \lambda_r = \sigma_r + i\omega_r \end{cases} \qquad \text{(wk NL)}$$

$$\sigma_0 = 0$$
$$\sigma_1 > 0$$

and $\underline{u}$ right eigenvector of L_0:

$$\begin{cases} \underline{\underline{L_0}}\,\underline{u} = \lambda_0 \underline{u} \\ \underline{\underline{L_0}}\,\overline{\underline{u}} = \bar{\lambda}_0 \overline{\underline{u}} \end{cases}$$

with $\underline{u}^*$ as left eigenvector:

$$\begin{cases} \underline{u}^* \underline{\underline{L_0}} = \lambda_0 \underline{u}^* \\ \overline{\underline{u}}^* \underline{\underline{L_0}} = \bar{\lambda}_0 \underline{u}^* \end{cases}$$

with normalization:

$$\underline{u}^{*} \cdot \underline{\bar{u}} = \underline{\bar{u}}^{*} \cdot \underline{u} = 0, \quad \underline{u}^{*} \cdot \underline{u} = \underline{\bar{u}}^{*} \cdot \underline{\bar{u}} = 1$$

and can write:

$$\lambda_0 = i\omega_0 = \underline{u}^{*} \cdot \underline{\underline{L_0}}\, \underline{u} \qquad [\ \ ] \cdot [\ \ ][\ \ ]$$

$(\sigma_0 = 0)$

$$\lambda_1 = \sigma_1 + i\omega_1 = \underline{u}^{*} \cdot \underline{\underline{L_1}}\, \underline{u}$$

and, define $\Rightarrow \epsilon \rightarrow$ measure of amplitude,

where $\epsilon^2 \chi \equiv \mu$

$\chi = \operatorname{sgn} \mu$

so $\underline{u} = \epsilon \underline{u_1} + \epsilon_2 \underline{u_2} + \cdots$

$$\underline{\underline{L}} = \underline{\underline{L_0}} + \epsilon^2 \chi\, \underline{\underline{L_1}} + \epsilon^4 \underline{\underline{L_2}} + \cdots$$

(rep $\langle u | L | u \rangle$)

and $M = M_0 + \epsilon^2 \chi\, M_1 + \cdots$

$$N = N_0 + \epsilon^2 \chi\, N_1 + \cdots$$

Further: as $\lambda_1 \sim O(\epsilon^2)$

rescale time τ as $\tau = \epsilon^2 t$

$\underline{u} = u(t, \tau)$

so

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau}$$

so finally:

$$\frac{d\underline{x}}{dt} = \underline{F}(\underline{x}, u) \Rightarrow$$

$$\frac{d\underline{y}}{dt} - L\underline{y} = M\underline{y}\underline{y} + N\underline{y}\underline{y}\underline{y}$$

and expanding:

$d/dt = \partial/\partial t + \epsilon^2 \partial/\partial \tau$

$L = L_0 + \epsilon^2 \chi L_1 + \epsilon^4 L_2 + \cdots$

$u = \epsilon u_1 + \epsilon^2 u_2 + \cdots$

plugging it all in $\Rightarrow$

$$\left(\frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial T} - L_0 - \epsilon^2 \chi L_1 - \cdots\right)\left(\epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + \cdots\right)$$

$$= \epsilon^2 M_0 \underline{u}_1 \underline{u}_1 + \epsilon^3 \left(2 M_0 \underline{u}_1 \underline{u}_2 + N_0 \underline{u}_1 \underline{u}_1 \underline{u}_1\right) + \cdots$$

so grinding order-by-order:

$$\left(\frac{\partial}{\partial t} - L_0\right) \underline{u}_1 = 0$$

$$\left(\frac{\partial}{\partial t} - L_0\right) \underline{u}_2 = M_0 \underline{u}_1 \underline{u}_1$$

$$\left(\frac{\partial}{\partial t} - L_0\right) \underline{u}_3 = -\left(\frac{\partial}{\partial T} - \chi L_1\right) \underline{u}_1 + \overset{\text{symmetrizing}}{2} M_0 \underline{u}_1 \underline{u}_2 + N_0 \underline{u}_1 \underline{u}_1 \underline{u}_1$$

etc

N.B.: RHS from lower order quantities.

Now have system of linear inhomogeneous equations:

$$\left(\frac{\partial}{\partial t} - L_0\right) \underline{U}_r = \underline{B}_r \qquad r = 1, 2, \ldots$$

but can note:

$$\boxed{\int_0^{2\pi/\omega_0} \underline{U}^* \cdot \underline{B}_r e^{-i\omega_0 t} = 0}$$

$\Rightarrow$ solvability condition (akin to removing resonant drive in Duffing)

since $\underline{B}_r = \left(\frac{\partial}{\partial t} - L_0\right) \underline{U}_r$

$\Rightarrow$

$$\int_0^{2\pi} \underline{U}^* \cdot \underline{B}_r e^{-i\omega_0 t} dt = \int_0^{2\pi/\omega_0} dt \left[\underline{U}^* \cdot \left(\frac{\partial}{\partial t} - \underline{\underline{L_0}}\right) U_r\right] e^{-i\omega_0 t}$$

$$= \int_0^{2\pi} dt \; \underline{U}^* \cdot \underline{U}^r \left(\underbrace{i\omega_0}_{\text{from ibp}} - \underbrace{i\omega_0}_{\text{eigenvalue}}\right) e^{-i\omega_0 t}$$

$\Rightarrow 0$ ✓

Observe: - $U_r \to$ periodic function of $\omega_0 t$

so

- $B_r(t, T) \to$ periodic in $\omega_0 t$

so $\quad \underline{B}_r(t,T) = \sum_{\ell=-\infty}^{+\infty} B_r^{\ell}(T)\, e^{i\ell\omega_0 t}$

so solvability $\Rightarrow$

$$\underline{U}^* \cdot B_r^{(1)}(T) = 0$$

(i.e. eigenvector projection of ω_0-coherent piece of RHS vanishes)

Now, consider: $\quad r = 1$

$$U_1(t,T) = \underbrace{W(T)}_{\text{complex amplitude}}\, \underline{U}\, e^{i\omega_0 t} + \text{c.c.}$$

→ neutral solution

→ $W(T)$ determined by $r = 3$ solvability

i.e.

$$\underline{U}^* \cdot \underline{B}_3^{(1)} = 0$$

Now:

→ $\quad \underline{U}^* \cdot \underline{B}_2^{(1)}(T) = 0$, trivially,

as B_2 contains only $\ell = 0$, $\ell = 2$ beats.
no phase coherent contribution

- BUT - must express $U^{(2)}$ in terms W to obtain equation for W via

$$\underline{U}^* \cdot \underline{B}_3^{(1)} = 0.$$

Now, can write:

$$u_2 = V_+ W^2 e^{2i\omega_0 t} + V_- \overline{W}^2 e^{-2i\omega_0 t} + V_0 |W|^2$$

$$+ \underbrace{V_0 u_1}_{\text{I.C.}}$$

and have:

$$\left(\frac{\partial}{\partial t} - L_0\right) \underline{u}_2 = M_0 \underline{u}_1 \underline{u}_1$$

$$\left(\frac{\partial}{\partial t} - L_0\right)\Big[V_+ W^2 e^{2i\omega_0 t} + V_- \overline{W}^2 e^{-2i\omega_0 t} + V_0 |W|^2$$

$$+ \overbrace{V_0 u_1}^{\text{free solution}} \Big] = M_0 \underline{u}_1 \underline{u}_1,$$

$$u_1(t,\tau) = W(\tau)\, \underline{y}\, e^{i\omega_0 t} + \text{c.c.}$$

so

$$\left(\frac{\partial}{\partial t} - L_0\right) V_+ W^2 e^{2i\omega_0 t} = \overset{\text{tensor!}}{M_0}\, W^2 \underline{y}\,\underline{y}$$

$$\Rightarrow \begin{cases} V_+ = -(L_0 - 2i\omega_0)^{-1} (M\, y\, y) \\ V_- = \overline{V_+} \end{cases}$$

and $\quad \overline{V}_0 = -2 L_0^{-1} U \overline{U}$. $\quad$ (2 zero frequency beats)

V_0 indeterminate.

Now, use $\underline{U}^* \cdot \underline{B}_3^{(1)} = 0$

with:

$$\begin{cases} \underline{U}_1 = W(T)\underline{U}\, e^{i\omega_0 T} + c.c. \\ \underline{U}_2 = V_+ W^2 e^{2i\omega_0 t} + V_- \overline{W}^2 e^{-2i\omega_0 t} + V_0 |W|^2 \\ \underline{B}_3 = -\left(\frac{\partial}{\partial T} - \chi L_1\right)\underline{U}_1 + 2 M_0\, \underline{U}_1 \underline{U}_2 \\ \qquad + N_0\, \underline{U}_1 \underline{U}_1 \underline{U}_1 \end{cases}$$

$$\Rightarrow \underline{B}_3^{(1)} = -\left(\frac{\partial}{\partial T} - \chi L_1\right) W \underline{U} + \Big(2 M_0\, U V_0 + 2 M_0 \overline{U} V_+ + 3 N_0 \underbrace{U U \overline{U}}_{\to\ 3 \text{ contrib.}}\Big) |W|^2 W$$

so $\quad U^* \cdot \underline{B}_3^{(1)} = 0 \quad \Rightarrow$

$$\boxed{\frac{\partial W}{\partial t} = \chi \lambda W - g|W|^2 W} \Rightarrow \boxed{\text{CGL equation}}$$

here;

$$g \equiv g' + i g'' \equiv -2\, \underline{U}^+ M \underline{U}\, \underline{V}_0 - 2\, \underline{U}^+ M_0 \overline{\underline{U}}\, \underline{V}_+ - 3\, \underline{U}^+ N \underline{U}\, \underline{U}\, \overline{\underline{U}}$$

$\Rightarrow$ recovers CGL equation !.

III.) Two Interacting Oscillators - ⓦ Weak Interaction.

Now, recall: $\underbrace{\text{single oscillator}}_{\omega_0}$ + $\underbrace{\text{forcing}}_{\omega}$

(incommensurate frequencies)

→ synchronization: oscillator locks to ω
⇒ single ω exhibited

→ quasiperiodicity: two incommensurate frequencies exhibited in ω spectrum.

Recall, in P.T.

$$\frac{d\phi}{dt} = \omega_0 + Q(\phi_0 + \omega_0 t, \underbrace{t}_{\omega t})$$

from phase piece of amplitude eqn → i.e. CGL

resonant terms dominate (DC forcing) ⇒ for $\omega \sim \omega_0$ (but $\omega \neq \omega_0$)

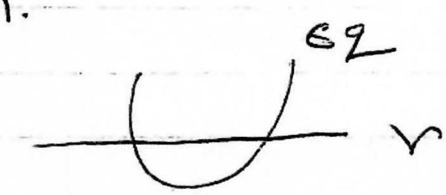
⇒ $\boxed{\psi \equiv \phi - \omega t}$ phase variable

and

$$\boxed{\frac{d\psi}{dt} = -\underbrace{\nu}_{\substack{\text{mismatch} \\ \omega - \omega_0}} + \underbrace{\epsilon q(\psi)}_{\text{forcing}}}$$

so fixed points $\Rightarrow \psi_{synch}$.

$$r = \epsilon q(\psi)$$



stability $\Rightarrow q'(\psi_s) < 0$ stable f.p. $\Rightarrow$ synch.
unstable $\Rightarrow q'(\psi_s) > 0$

i.e at ψ_{synch} $\psi = \phi - \omega t = \psi_{synch}$
$\phi = \psi_{synch} + \omega t$.

$\rightarrow$ 1) Synchronization is a bifurcation/transition

2) Synch possible for:
$\epsilon q_{min} < r < \epsilon q_{max}$

otherwise incommensurate frequency

3) outside synchronization region

$$\phi = \omega t + \psi(t) \qquad \rightarrow \text{q.p. motion}$$

4) beat frequency $\Omega_\psi = 2\pi / T_\psi$

$$T_\psi = \left| \int_0^{2\pi} d\psi \,/\, (\epsilon q(\psi) - r) \right| \rightarrow \text{beat period}$$

$\rightarrow$ effective difference between oscillator frequency and external force frequency.

5) Near $q_{\substack{max \\ min}}$ → ψ lingers

i.e. $\Omega_\psi \sim (r - r_m)^{1/2}$

⇒ beat frequency slows near synch,

system spends long time in near synchrony, interspersed with brief periods of phase slippage.

Slip longer than ω^{-1}.

→ Amplitude Equations: Coupled Oscillators

Consider 2 weakly nonlinear oscillators:

$$\ddot{x}_1 + \omega_1^2 x_1 = f_1(x_1, \dot{x}_1) + D_1(x_2 - x_1) + B_1(\dot{x}_2 - \dot{x}_1)$$

$$\ddot{x}_2 + \omega_2^2 x_2 = f_2(x_2, \dot{x}_2) + D_2(x_1 - x_2) + B_2(\dot{x}_1 - \dot{x}_2)$$

- linear coupling
- difference coupling ⟷ "diffusive" (anticipates phase diffusion)

but also can be... ⟷ "direct" coupling

i.e. $RHS_1 = D_1 x_2 + B_1 x_2$

Aim: Link between structure of coupling and macro-phenomena (i.e. oscillation death)

As before, $(x, y)_{1,2} = \left(\begin{matrix} 1/2\, A_{1,2}(t)\, e^{i\omega t} + c.c. \\ i\omega\,(\quad) \end{matrix} \right)$

⇒ amplitude equations via averaging ⇒

$$\begin{cases} \dot{A}_1 = -i\,\Delta_1 A_1 + \mu_1 A_1 - (\gamma_1 + i\alpha_1)|A_1|^2 A_1 + (\beta_1 + i\delta_1)(A_2 - A_1) \\ \dot{A}_2 = -i\,\Delta_2 A_2 + \mu_2 A_2 - (\gamma_2 + i\alpha_2)|A_2|^2 A_2 + (\beta_2 + i\delta_2)(A_1 - A_2) \end{cases}$$

$\Delta_{1,2} \to$ reactive (ω effect)

i.e. $\text{Coupling}_1 = (\beta_1 + i\delta_1)(A_2 - A_1)$

$\hookrightarrow B_{1,2} \to$ dissipative

$\text{Coupling}_2 = (\beta_2 + i\delta_2)(A_1 - A_2)$

$\Delta_{1,2} = \omega_{1,2} - \omega \quad \to$ mis-match.

Now, to save algebra:

- $A_{1,2} = R_{1,2}\, e^{i\phi_{1,2}}$ (Amplitude Phase Rep.)

- $\psi = \phi_2 - \phi_1$ (via difference coupling)

$\Rightarrow$

$$\partial R_1/\partial t = \mu_1 R_1 (1 - \gamma_1 R_1^2) + \beta_1 (R_2 \cos\psi - R_1) - \delta_1 R_2 \sin\psi$$

$$\partial R_2/\partial t = \mu_2 R_2 (1 - \gamma_2 R_2^2) + \beta_2 (R_1 \cos\psi - R_2) + \delta_2 R_1 \sin\psi$$

$$\partial \psi/\partial t = -\nu + \mu_1 \alpha_1 R_1^2 - \mu_2 \alpha_2 R_2^2 + \left(\delta_2 \frac{R_1}{R_2} - \delta_1 \frac{R_2}{R_1}\right)\cos\psi + \delta_1 - \delta_2 - \left(\beta_1 \frac{R_2}{R_1} + \beta_2 \frac{R_1}{R_2}\right)\sin\psi$$

Further: clears system
$$\begin{cases} \mu_1 = \mu_2 = \mu \\ t \to t/\mu \\ A \to A/(\gamma/\mu)^{1/2} \\ \beta_0\ \delta \to \text{normalized to } \mu \\ \alpha \to \text{normalized to } \gamma/\mu \end{cases}$$

$\Rightarrow$

$$\begin{cases} \dot{R}_1 = R_1(1-R_1^2) + \overset{*}{\beta}(R_2\cos\psi - R_1) - \overset{*}{\delta} R_2 \sin\psi \\ \dot{R}_2 = R_2(1-R_2^2) + \overset{*}{\beta}(R_1\cos\psi - R_2) + \overset{*}{\delta} R_1 \sin\psi \\ \dot{\psi} = -\nu + \overset{*}{\alpha}(R_1^2 - R_2^2) + \overset{*}{\delta}\left(-\frac{R_2}{R_1} + \frac{R_1}{R_2}\right)\cos\psi \\ \qquad - \overset{*}{\beta}\left(\frac{R_2}{R_1} + \frac{R_1}{R_2}\right)\sin\psi \end{cases}$$

Phase and 2 Amplitude System:

$\alpha \to$ NL frequency shift
$\Leftrightarrow$ $\alpha = 0$ "isochronous" (new use)

$\nu \to$ frequency detuning

$\delta \to$ reactive coupling
$\beta \to$ dissipative coupling

Now, consider phenomena exhibited by the system

a.) oscillation death / quenching

b.) attractive / repulsive interaction

a.) Oscillation Death

→ large β, γ ⇒ $R_1 = R_2 = 0$ becomes stable.

∴

→ oscillations die.

To see: – $\delta \equiv 0$ dissipative coupling only (via β)
– $\omega \equiv (\omega_1 + \omega_2)/2$ so
$\Delta_1 = -\Delta_2 = \Delta$

and obtain, for amplitude equation:

$$\dot{A}_1 = (i\Delta + \mu) A_1 + \beta (A_2 - A_1) + \cancel{NL}$$

$$\dot{A}_2 = (-i\Delta + \mu) A_2 + \beta (A_1 - A_2) + \cancel{NL}$$

i.e. perturb about $A_1 = A_2 = 0$

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_{1,0} \\ A_{2,0} \end{pmatrix} e^{\lambda t}$$

$$\Rightarrow \boxed{\lambda = \mu - \beta \pm \sqrt{\beta^2 - \Delta^2}}$$

Need: $\left[\begin{array}{l} \lambda < 0 \\ \mu < \beta \quad \text{and} \quad \beta < (\mu^2 + \Delta^2)/2\mu. \end{array}\right.$

Key: ① $\underline{\mu < \beta}$

② $\underline{\beta < \frac{\Delta^2}{2\mu} + \ldots}$

① → "diffusive" coupling brings additional "difference" dissipation to each oscillator, i.e. each 'drags other down)'

② → detuning is large enough so forcing from other oscillator can't excite.

b.) Attractive / Repulsive Interaction

- reduce to phase description
- derive directly; for β, δ small

excursion

$\downarrow$

so $\quad R_{1,2} \approx 1 + r_{1,2}$ (perturb about oscillator)

$r_{1,2} \ll 1$

$\Rightarrow$ plugging in to $\dot{R}_1$, $\dot{R}_2$ and linearizing $\Rightarrow$

$$\dot{r}_1 = -2r_1 + \beta(\cos\psi - 1) - \delta\sin\psi$$

$$\dot{r}_2 = -2r_2 + \beta(\cos\psi - 1) + \delta\sin\psi$$

strong damping $\Rightarrow$ $\dot{r}_1 = \dot{r}_2 = 0$

$$\therefore \quad r_{1,2} = \frac{\beta}{2}(\cos\psi - 1) \mp \frac{\delta}{2}\sin\psi$$

$$R_{1,2} = 1 + r_{1,2}$$

and plugging onto phase equation:

$\psi = \phi_2 - \phi_1$

$$\boxed{\dot{\psi} = -\nu - 2(\beta + \alpha\delta)\sin\psi}$$

phase dynamics equation!

Attractive + Repulsive Interaction

Aside: if $g(\psi) = \sin\psi$

$$\frac{d\psi}{dt} = -\nu + \epsilon \sin\psi$$

so ① $\epsilon < 0 \Rightarrow$ stable f.p. (ψ_{synch}) on $-\pi/2 < \psi < \pi/2$

i.e. $$\frac{d\delta\psi}{dt} = \epsilon \cos\psi_s \, \delta\psi$$

so $\nu \to 0$ $\quad \psi_s = 0$

stable phase difference zero

$\Rightarrow$ phases "attract"

② $\epsilon > 0 \Rightarrow$ stable f.p. (ψ_{synch}) on $\pi/2 < \psi < 3\pi/2$

so $\nu \to 0$ $\quad \psi_s = \pi$

stable phase difference π

$\Rightarrow$ phases "repel"

Now, clear from before:

if $\gamma = 0$

$\beta + \alpha\delta > 0 \Rightarrow \psi = 0$ is stable ψ_s.

$\Rightarrow$ "attraction"

$\beta + \alpha\delta < 0 \Rightarrow \psi = \pi$ is stable ψ_s

$\Rightarrow$ "repulsion"

To interpret:

$\beta \rightarrow B_{1,2} \rightarrow$ dissipative coupling

$\delta \rightarrow D_{1,2} \rightarrow$ reactive $\leftrightarrow$ shift eigenfrequencies

$\therefore$ β - dissipative coupling
- drives 2 oscillators to more homogeneous regime

$\Rightarrow$ 'toward' synchronization via drag on each other

$\Rightarrow$ attraction.

δ - reactive coupling

$\Rightarrow$ no effect on isochronous oscillators ($\alpha = 0$)

$\Rightarrow$ non-isochronous oscillators $\Rightarrow$

attractive or repulsive, depending on α's sign.