PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #6 SOLUTIONS

(1) Consider the one-dimensional Ising model with next-nearest neighbor interactions,

$$\hat{H} = -J\sum_n \sigma_n \sigma_{n+1} - K\sum_n \sigma_n \sigma_{n+2} \,,$$

on a ring with N sites, where N is even. By considering consecutive pairs of sites, show that the partition function may be written in the form $Z = \mathrm{Tr}(R^{N/2})$, where R is a 4×4 transfer matrix. Find R. *Hint:* It may be useful to think of the system as a railroad trestle, depicted in Fig. 2, with Hamiltonian

$$\hat{H} = -\sum_j \Big[J\sigma_j \mu_j + J\mu_j \sigma_{j+1} + K\sigma_j \sigma_{j+1} + K\mu_j \mu_{j+1}\Big] \,.$$

Then $R = R_{(\sigma_j\mu_j),(\sigma_{j+1}\mu_{j+1})}$, with $(\sigma\mu)$ a composite index which takes one of four possible values $(++)$, $(+-)$, $(-+)$, or $(--)$.

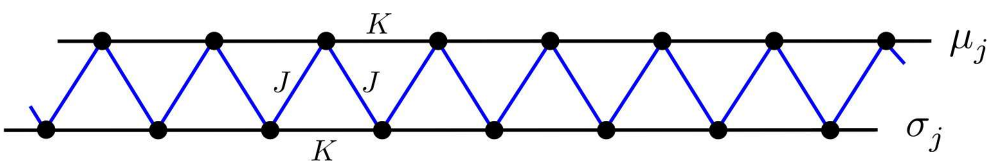

Figure 1: Railroad trestle representation of next-nearest neighbor chain.

Solution :

The transfer matrix can be read off from the Hamiltonian:

$$R_{(\sigma\mu),(\sigma'\mu')} = e^{\beta J \mu(\sigma+\sigma')}\, e^{\beta K(\sigma\sigma' + \mu\mu')} \,.$$

Expressed as a matrix of rank four, with rows and columns corresponding to $\{++, +-, --, -+\}$, we have

$$R = \begin{pmatrix} e^{2\beta J} e^{2\beta K} & e^{2\beta J} & e^{-2\beta K} & 1 \\ e^{-2\beta J} & e^{-2\beta J} e^{2\beta K} & 1 & e^{-2\beta K} \\ e^{-2\beta K} & 1 & e^{2\beta J} e^{2\beta K} & e^{-2\beta J} \\ 1 & e^{-2\beta K} & e^{2\beta J} & e^{-2\beta J} e^{2\beta K} \end{pmatrix} .$$

Querying WolframAlpha for the eigenvalues, we find

$$\begin{aligned}
\lambda_1 &= \tfrac{1}{2}\Big[uv + (1+u^{-1})\sqrt{u^2v^2 - 2uv^2 + 4u + v^2} + 2v^{-1} + u^{-1}v \Big] \\
\lambda_2 &= \tfrac{1}{2}\Big[uv + (1-u^{-1})\sqrt{u^2v^2 + 2uv^2 - 4u + v^2} - 2v^{-1} + u^{-1}v \Big] \\
\lambda_3 &= \tfrac{1}{2}\Big[uv - (1+u^{-1})\sqrt{u^2v^2 - 2uv^2 + 4u + v^2} + 2v^{-1} + u^{-1}v \Big] \\
\lambda_4 &= \tfrac{1}{2}\Big[uv - (1-u^{-1})\sqrt{u^2v^2 + 2uv^2 - 4u + v^2} - 2v^{-1} + u^{-1}v \Big] \quad ,
\end{aligned}$$

where $u = e^{2\beta J}$ and $v = e^{2\beta K}$. The partition function on a ring of N sites, with N even, is

$$Z = \mathsf{Tr}\left(R^{N/2}\right) = \lambda_1^{N/2} + \lambda_2^{N/2} + \lambda_3^{N/2} + \lambda_4^{N/2} \,.$$

It may seem a happy accident that the nonsymmetric matrix R has all real eigenvalues. However, note that

$$\widetilde{R} \equiv \varSigma R \varSigma = W X$$

for

$$\varSigma = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad , \quad W = \begin{pmatrix} \cosh(2\beta J) & \sinh(2\beta J) & e^{-2\beta K} & 0 \\ \sinh(2\beta J) & \cosh(2\beta J) & 0 & -e^{-2\beta K} \\ e^{-2\beta K} & 0 & \cosh(2\beta J) & \sinh(2\beta J) \\ 0 & -e^{-2\beta K} & \sinh(2\beta J) & \cosh(2\beta J) \end{pmatrix}$$

and

$$X = \begin{pmatrix} e^{2\beta K}+1 & 0 & 0 & 0 \\ 0 & e^{2\beta K}-1 & 0 & 0 \\ 0 & 0 & e^{2\beta K}+1 & 0 \\ 0 & 0 & 0 & e^{2\beta K}-1 \end{pmatrix} \quad .$$

Note that $\varSigma^2 = 1$, hence $\varSigma^{-1} = \varSigma$. Thus,

$$R = \varSigma W X \varSigma = \varSigma X^{-1/2}\big(X^{1/2} W X^{1/2}\big) X^{1/2} \varSigma$$

and we find R is related by similarity transformation to the symmetric matrix $X^{1/2} W X^{1/2}$. Here we assume $K > 0$ so $e^{2\beta K} \pm 1 > 0$ and we may take the square root of X. If $K < 0$ one readily obtains a similar construction.

C. Aganze notes that we may define[1] $\nu_n \equiv \sigma_n \sigma_{n+1}$, in which case $\nu_n \nu_{n+1} = \sigma_n \sigma_{n+2}$, and our Hamiltonian becomes

$$\hat{H} = -K \sum_n \nu_n \nu_{n+1} - J \sum_n \nu_n \quad ,$$

i.e. a nearest neighbor Ising model with ferromagnetic exchange K and a field $h = J$. From the lecture notes, we know that the eigenvalues of the transfer matrix are given by

$$\begin{aligned} \varLambda_\pm &= e^{\beta K} \cosh(\beta J) \pm \sqrt{e^{2\beta K} \sinh^2(\beta J) + e^{-2\beta K}} \\ &= \tfrac{1}{2}(uv)^{-1/2} \left\{ (1+u)v \pm \sqrt{(u-1)^2\, v^2 + 4u} \right\} \quad . \end{aligned}$$

It is now easy to show that $\varLambda_+^2 = \lambda_1$, whence the equality of the partition functions in the thermodynamic limit $N \to \infty$ is manifest.: $\varLambda_+^N = \lambda_1^{N/2}$. Note that on a ring, there is a constraint $\prod_{n=1}^N \nu_n = 1$ which must be satisfied.

[1]See also §6.2.4 of the Lecture Notes.

(2) For each of the cluster diagrams in Fig. 2, find the symmetry factor s_γ and write an expression for the cluster integral b_γ.

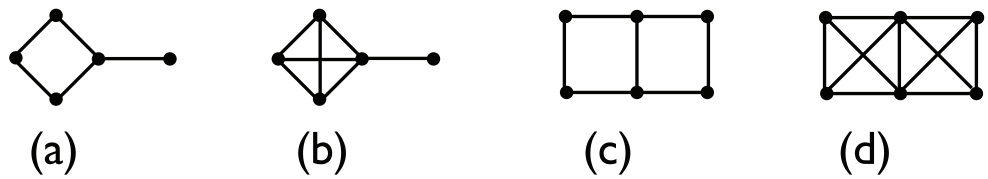

Figure 2: Cluster diagrams for problem 2.

Solution :

Choose labels as in Fig. 3, and set $\boldsymbol{x}_{n_\gamma} \equiv 0$ to cancel out the volume factor in the definition of b_γ.

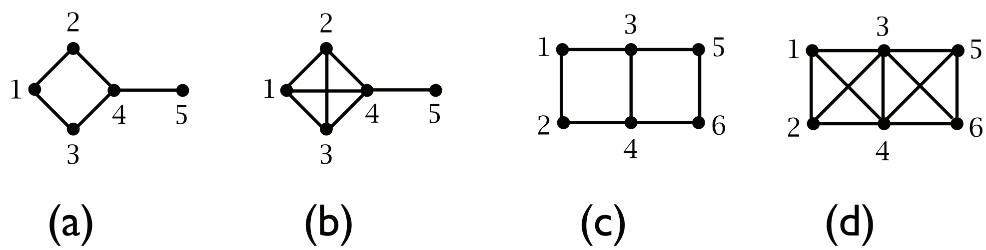

Figure 3: Labeled cluster diagrams.

(a) The symmetry factor is $s_\gamma = 2$, so

$$b_\gamma = \tfrac{1}{2}\int\! d^d\!x_1 \!\int\! d^d\!x_2 \!\int\! d^d\!x_3 \!\int\! d^d\!x_4 \; f(r_{12})\, f(r_{13})\, f(r_{24})\, f(r_{34})\, f(r_4) \,.$$

(b) Sites 1, 2, and 3 may be permuted in any way, so the symmetry factor is $s_\gamma = 6$. We then have

$$b_\gamma = \tfrac{1}{6}\int\! d^d\!x_1 \!\int\! d^d\!x_2 \!\int\! d^d\!x_3 \!\int\! d^d\!x_4 \; f(r_{12})\, f(r_{13})\, f(r_{24})\, f(r_{34})\, f(r_{14})\, f(r_{23})\, f(r_4) \,.$$

(c) The diagram is symmetric under reflections in two axes, hence $s_\gamma = 4$. We then have

$$b_\gamma = \tfrac{1}{4}\int\! d^d\!x_1 \!\int\! d^d\!x_2 \!\int\! d^d\!x_3 \!\int\! d^d\!x_4 \!\int\! d^d\!x_5 \; f(r_{12})\, f(r_{13})\, f(r_{24})\, f(r_{34})\, f(r_{35})\, f(r_4)\, f(r_5) \,.$$

(d) The diagram is symmetric with respect to the permutations (12), (34), (56), and $(15)(26)$. Thus, $s_\gamma = 2^4 = 16$. We then have

$$b_\gamma = \tfrac{1}{16}\int\! d^d\!x_1 \!\int\! d^d\!x_2 \!\int\! d^d\!x_3 \!\int\! d^d\!x_4 \!\int\! d^d\!x_5 \; f(r_{12})\, f(r_{13})\, f(r_{14})\, f(r_{23})\, f(r_{24})\, f(r_{34})\, f(r_{35})\, f(r_{45})\, f(r_3)\, f(r_4)\, f(r_5) \,.$$

(3) Compute the partition function for the one-dimensional Tonks gas of hard rods of length a on a ring of circumference L. This is slightly tricky, so here are some hints. Once again, assume a particular ordering so that $x_1 < x_2 < \cdots < x_N$. Due to translational invariance, we can define the positions of particles $\{2, \ldots, N\}$ relative to that of particle 1, which we initially place at $x_1 = 0$. Then periodicity means that $x_N \le L - a$, and in general one then has

$$x_{j-1} + a \le x_j \le L - (N - j + 1)a \ .$$

Now integrate over $\{x_2, \ldots, x_N\}$ subject to these constraints. Finally, one does the x_1 integral, which is over the entire ring, but which must be corrected to eliminate overcounting from cyclic permutations. How many cyclic permutations are there?

Solution :

There are N cyclic permutations, hence the last x_1 integral yields L/N, and

$$Z(T, L, N) = \lambda_T^{-N} \, \frac{L}{N} \int\limits_{a}^{Y_2} \! dx_2 \int\limits_{x_2+a}^{Y_3} \! dx_3 \, \cdots \int\limits_{x_{N-1}+a}^{Y_N} \! dx_N = \frac{L(L - Na)^{N-1} \lambda_T^{-N}}{N!} \ .$$