

PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #5 SOLUTIONS

(1) Using the argument we used in class and in §5.5.3 of the notes, predict the surface temperatures of the remaining planets in our solar system. In each case, compare your answers with the most reliable source you can find. In cases where there are discrepancies, try to come up with a convincing excuse.

Solution :

Relevant planetary data are available from Wikipedia. According to the derivation in the notes, we have

$$T = \left(\frac{R_{\odot}}{2a} \right)^{1/2} T_{\odot},$$

where $R_{\odot} = 6.96 \times 10^5$ km and $T_{\odot} = 5780$ K. From this equation and the reported values for a for each planet, we obtain the following table:

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
a (10^8 km)	0.576	1.08	1.50	2.28	7.78	14.3	28.7	45.0	59.1
$T_{\text{surf}}^{\text{obs}}$ (K)	340*	735 [†]	288 [†]	210	112	84	53	55	44
$T_{\text{surf}}^{\text{pred}}$ (K)	448	327	278	226	122	89.1	63.6	50.8	44.3

Table 1: Planetary data from GSU web site and from Wikipedia. Observed temperatures are averages. * mean equatorial temperature. [†] mean temperature below cloud cover.

Note that we have included Pluto, because since my childhood Pluto has always been the ninth planet to me. We see that our simple formula works out quite well except for Mercury and Venus. Mercury, being so close to the sun, has enormous temperature fluctuations as a function of location. Venus has a whopping greenhouse effect.

(2) Read carefully the new and improved §5.6.4 of the lecture notes (“Melting and the Lindemann criterion”). Using the data in Table 5.1, and looking up the atomic mass and lattice constant of tantalum (Ta), find the temperature T_L where the Lindemann criterion predicts Ta should melt.

Solution :

One finds the mass of tantalum is $M = 181$ amu, and the lattice constant is $a = 3.30$ Å. Thus,

$$\Theta^* = \frac{109 \text{ K}}{M[\text{amu}](a[\text{Å}])^2} = 55.3 \text{ mK}.$$

From the table in the lecture notes, the Debye temperature is $\Theta_D = 246$ K and the melting

point is $T_{\text{melt}} = 2996 \text{ K}$. The Lindemann temperature is

$$T_L = \left(\frac{\eta^2 \Theta_D}{\Theta^*} - 1 \right) \frac{\Theta_D}{4} = 2674 \text{ K} ,$$

where $\eta = 0.10$. Close enough for government work.

(3) Consider a two-dimensional gas of fermions which obey the dispersion relation

$$\varepsilon(\mathbf{k}) = \varepsilon_0 \left((k_x^2 + k_y^2) a^2 + \frac{1}{2} (k_x^4 + k_y^4) a^4 \right) .$$

Sketch, on the same plot, the Fermi surfaces for $\varepsilon_F = 0.1 \varepsilon_0$, $\varepsilon_F = \varepsilon_0$, and $\varepsilon_F = 10 \varepsilon_0$.

Solution:

It is convenient to adimensionalize, writing

$$x \equiv k_x a \quad , \quad y \equiv k_y a \quad , \quad \nu \equiv \frac{\varepsilon}{\varepsilon_0} . \quad (1)$$

Then the equation for the Fermi surface becomes

$$x^2 + y^2 + \frac{1}{2} x^4 + \frac{1}{2} y^4 = \nu . \quad (2)$$

In other words, we are interested in the *level sets* of the function $\nu(x, y) \equiv x^2 + y^2 + \frac{1}{2} x^4 + \frac{1}{2} y^4$. When ν is small, we can ignore the quartic terms, and we have an isotropic dispersion, with $\nu = x^2 + y^2$. *I.e.* we can write $x = \nu^{1/2} \cos \theta$ and $y = \nu^{1/2} \sin \theta$. The quartic terms give a contribution of order ν^4 , which is vanishingly small compared with the quadratic term in the $\nu \rightarrow 0$ limit. When $\nu \sim \mathcal{O}(1)$, the quadratic and quartic terms in the dispersion are of the same order of magnitude, and the continuous $O(2)$ symmetry, namely the symmetry under rotation by any angle, is replaced by a discrete symmetry group, which is the group of the square, known as C_{4v} in group theory parlance. This group has eight elements:

$$\{\mathbb{I}, R, R^2, R^3, \sigma, \sigma R, \sigma R^2, \sigma R^3\} \quad (3)$$

Here R is the operation of counterclockwise rotation by 90° , sending (x, y) to $(-y, x)$, and σ is reflection in the y -axis, which sends (x, y) to $(-x, y)$. One can check that the function $\nu(x, y)$ is invariant under any of these eight operations from C_{4v} .

Explicitly, we can set $y = 0$ and solve the resulting quadratic equation in x^2 to obtain the maximum value of x , which we call $a(\nu)$. One finds

$$\frac{1}{2} x^4 + x^2 - \nu = 0 \quad \implies \quad a = \sqrt{\sqrt{1 + 2\nu} - 1} .$$

So long as $x \in \{-a, a\}$, we can solve for $y(x)$:

$$y(x) = \pm \sqrt{\sqrt{1 + 2\nu - 2x^2 - x^4} - 1} .$$

A sketch of the level sets, showing the evolution from an isotropic (*i.e.* circular) Fermi surface at small ν , to surfaces with discrete symmetries, is shown in fig. 1.

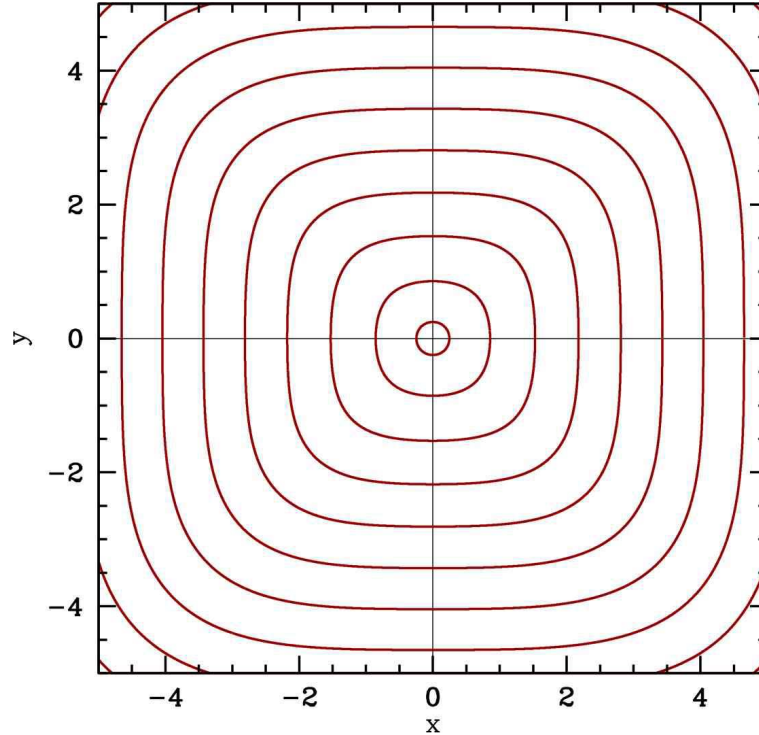


Figure 1: Level sets of the function $\nu(x, y) = x^2 + y^2 + \frac{1}{2}x^4 + \frac{1}{2}y^4$ for $\nu = (\frac{1}{2}n)^4$, with positive integer n .

(4) Show that the chemical potential of a three-dimensional ideal nonrelativistic Fermi gas is given by

$$\mu(n, T) = \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\varepsilon_F} \right)^2 - \frac{\pi^4}{80} \left(\frac{k_B T}{\varepsilon_F} \right)^4 + \dots \right]$$

and the average energy per particle is

$$\frac{E}{N} = \frac{3}{5} \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\varepsilon_F} \right)^2 - \frac{\pi^4}{16} \left(\frac{k_B T}{\varepsilon_F} \right)^4 + \dots \right],$$

where $\mu_0(n)$ is the Fermi energy at $T = 0$. Compute the heat capacity $C_V(T)$ to terms of order T^3 . How does the T^3 contribution to the electronic heat capacity compare with the contribution from phonons?

Solution:

From the Sommerfeld expansion we have

$$\begin{aligned} \int_{-\infty}^{\infty} d\varepsilon \phi(\varepsilon) f(\varepsilon - \mu) &= \int_{-\infty}^{\mu} d\varepsilon \phi(\varepsilon) + \frac{\pi^2}{6} (k_B T)^2 \phi'(\mu) + \frac{7\pi^4}{360} (k_B T)^4 \phi'''(\mu) + \mathcal{O}(T^6) \\ &= \left\{ 1 + \frac{\pi^2}{6} (k_B T)^2 \frac{d^2}{d\mu^2} + \frac{7\pi^4}{360} (k_B T)^2 \frac{d^4}{d\mu^4} + \mathcal{O}(T^6) \right\} \Phi(\mu), \end{aligned}$$

where $\phi(\varepsilon) = \Phi'(\varepsilon)$. Let's work this out to second order in T^2 for the case $\phi(\varepsilon) = g(\varepsilon)$. The integral then gives the overall density n . We write $\mu = \varepsilon_F + \delta\mu$ and expand the RHS to second order in $\delta\mu$. This yields

$$\begin{aligned} n &= \int_{-\infty}^{\varepsilon_F} d\varepsilon g(\varepsilon) + g(\varepsilon_F) \delta\mu + \frac{1}{2} g'(\varepsilon_F) (\delta\mu)^2 + \dots \\ &\quad + \frac{\pi^2}{6} (k_B T)^2 g'(\varepsilon_F) + \frac{\pi^2}{6} (k_B T)^2 g''(\varepsilon_F) \delta\mu + \dots \\ &\quad + \frac{7\pi^4}{360} (k_B T)^4 g'''(\varepsilon_F) + \dots \end{aligned}$$

RHS of the first line above comes from expanding the integral in the first term on the RHS of the previous equation to second order in $\delta\mu$. The subsequent lines come from the expansions of the second and third terms on the RHS of the previous equation, respectively. We expand out to the necessary order in each case. From this equation we thus obtain

$$\delta\mu = -\frac{\pi^2}{6} (k_B T)^2 \frac{g'(\varepsilon_F)}{g(\varepsilon_F)} - \frac{\pi^4}{36} (k_B T)^4 \cdot \left[\frac{1}{2} \left(\frac{g'(\varepsilon_F)}{g(\varepsilon_F)} \right)^3 - \frac{g'(\varepsilon_F) g''(\varepsilon_F)}{g^2(\varepsilon_F)} + \frac{7}{10} \frac{g'''(\varepsilon_F)}{g(\varepsilon_F)} \right] + \dots$$

If we assume that $g(\varepsilon)$ is a homogeneous function with $g(\varepsilon) \propto \varepsilon^\alpha$, then find

$$\delta\mu = -\frac{\alpha \pi^2}{6} \frac{(k_B T)^2}{\varepsilon_F} - \alpha(\alpha - 2)(2\alpha - 7) \frac{\pi^4}{360} \frac{(k_B T)^4}{\varepsilon_F^3} + \mathcal{O}(T^6).$$

Substituting $\alpha = \frac{1}{2}$, as is appropriate for three-dimensional ballistic fermions, we obtain

$$\delta\mu = -\frac{\pi^2}{12} \frac{(k_B T)^2}{\varepsilon_F} - \frac{\pi^4}{80} \frac{(k_B T)^4}{\varepsilon_F^3} + \mathcal{O}(T^6),$$

which is the desired result.

The result for the energy is tedious to carry out by hand, but is rather straightforward using a symbolic manipulation program such as Mathematica or Maple. We assume the density of states is of the form $g(\varepsilon) = C\varepsilon^\alpha$. Then from the Sommerfeld expansion we have

$$\begin{aligned} \frac{E}{V} &= \frac{C\mu^{\alpha+2}}{\alpha+2} \left\{ 1 + \alpha(\alpha+1) \frac{\pi^2}{6} \left(\frac{k_B T}{\mu} \right)^2 + (\alpha-2)(\alpha-1)\alpha(\alpha+1) \frac{7\pi^4}{360} \left(\frac{k_B T}{\mu} \right)^4 + \dots \right\} \\ \frac{N}{V} &= \frac{C\mu^{\alpha+1}}{\alpha+2} \left\{ 1 + (\alpha-1)\alpha \frac{\pi^2}{6} \left(\frac{k_B T}{\mu} \right)^2 + (\alpha-3)(\alpha-2)(\alpha-1)\alpha \frac{7\pi^4}{360} \left(\frac{k_B T}{\mu} \right)^4 + \dots \right\} \end{aligned}$$

Carefully taking the ratio and evaluating to order T^4 , we find

$$\frac{E}{N} = \left(\frac{\alpha + 1}{\alpha + 2} \right) \mu \cdot \left\{ 1 + (\alpha + 1) \frac{\pi^2}{3} \left(\frac{k_B T}{\mu} \right)^2 + \alpha(\alpha + 1)(\alpha - 6) \frac{\pi^4}{45} \left(\frac{k_B T}{\mu} \right)^4 + \dots \right\}.$$

Unfortunately we're not quite done, since we now must expand μ in a power series in T , invoking our previous result. Working this out (by hand!), I obtain

$$\frac{E}{N} = \left(\frac{\alpha + 1}{\alpha + 2} \right) \varepsilon_F \cdot \left\{ 1 + (\alpha + 2) \frac{\pi^2}{6} \left(\frac{k_B T}{\varepsilon_F} \right)^2 + \alpha(\alpha + 2)(2\alpha - 7) \frac{\pi^4}{120} \left(\frac{k_B T}{\varepsilon_F} \right)^4 + \mathcal{O}(T^6) \right\}.$$

Setting $\alpha = \frac{1}{2}$ we have $\frac{1}{6}(\alpha + 2) = \frac{5}{12}$ and $\frac{1}{120}\alpha(\alpha + 2)(2\alpha - 7) = -\frac{1}{16}$, as indicated in the statement of the problem. Our formula holds for general α , so we can find the result for $d = 2$ by setting $\alpha = 0$.

The heat capacity is

$$\begin{aligned} C_{V,N} &= \left(\frac{\partial E}{\partial T} \right)_{V,N} \\ &= Nk_B \left\{ (\alpha + 1) \frac{\pi^2}{3} \left(\frac{k_B T}{\varepsilon_F} \right) + \alpha(\alpha + 1)(2\alpha - 7) \frac{\pi^4}{30} \left(\frac{k_B T}{\varepsilon_F} \right)^3 + \mathcal{O}(T^5) \right\}. \end{aligned}$$

In $d = 3$, with $\alpha = \frac{1}{2}$, the order T^3 term is $-\frac{3}{20}\pi^4(k_B T/\varepsilon_F)^3 Nk_B$. The low temperature phonon contribution is $C_V^{(\text{phonon})} = \frac{12}{5}\pi^4 (T/\Theta_D)^3 Nk_B$, where Θ_D is the Debye temperature. The ratio is

$$\frac{\Delta C_V^{(\text{elec})}}{C_V^{(\text{phonon})}} = -\frac{1}{16} \left(\frac{\Theta_D}{T_F} \right)^3.$$

Since Θ_D is typically hundreds of K while T_F is tens of thousands of K, this ratio is on the order of 10^{-7} .

(5) Consider a three-dimensional Bose gas of particles which have two internal polarization states, labeled by $\sigma = \pm 1$. The single particle energies are given by

$$\varepsilon(\mathbf{p}, \sigma) = \frac{\mathbf{p}^2}{2m} + \sigma \Delta,$$

where $\Delta > 0$.

- Find the density of states per unit volume $g(\varepsilon)$.
- Find an implicit expression for the condensation temperature $T_c(n, \Delta)$. When $\Delta \rightarrow \infty$, your expression should reduce to the familiar one derived in class.
- When $\Delta = \infty$, the condensation temperature should agree with the familiar result for three-dimensional Bose condensation. Assuming $\Delta \gg k_B T_c(n, \Delta = \infty)$, find analytically the leading order difference $T_c(n, \Delta) - T_c(n, \Delta = \infty)$.

Solution :

(a) Let $g_0(\varepsilon)$ be the DOS per unit volume for the case $\Delta = 0$. Then

$$g_0(\varepsilon) d\varepsilon = \frac{d^3k}{(2\pi)^3} = \frac{k^2 dk}{2\pi^2} \Rightarrow g_0(\varepsilon) = \frac{\sqrt{2m^3}}{2\pi^2 \hbar^3} \varepsilon^{1/2} \Theta(\varepsilon).$$

For finite Δ , the single particle energies are shifted uniformly by $\pm\Delta$ for the $\sigma = \pm 1$ states, hence

$$g(\varepsilon) = g_0(\varepsilon + \Delta) + g_0(\varepsilon - \Delta).$$

(b) For Bose statistics, we have in the uncondensed phase,

$$\begin{aligned} n &= \int_{-\infty}^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{(\varepsilon-\mu)/k_B T} - 1} \\ &= \text{Li}_{3/2}(e^{(\mu+\Delta)/k_B T}) \lambda_T^{-3} + \text{Li}_{3/2}(e^{(\mu-\Delta)/k_B T}) \lambda_T^{-3}. \end{aligned}$$

In the condensed phase, $\mu = -\Delta - \mathcal{O}(N^{-1})$ is pinned just below the lowest single particle energy, which occurs for $\mathbf{k} = \mathbf{p}/\hbar = 0$ and $\sigma = -1$. We then have

$$n = n_0 + \zeta(3/2) \lambda_T^{-3} + \text{Li}_{3/2}(e^{-2\Delta/k_B T}) \lambda_T^{-3}.$$

To find the critical temperature, set $n_0 = 0$ and $\mu = -\Delta$:

$$n = \zeta(3/2) \lambda_{T_c}^{-3} + \text{Li}_{3/2}(e^{-2\Delta/k_B T_c}) \lambda_{T_c}^{-3}.$$

This is a nonlinear and implicit equation for $T_c(n, \Delta)$. When $\Delta = \infty$, we have

$$k_B T_c^\infty(n) = \frac{2\pi\hbar^2}{m} \left(\frac{n}{\zeta(3/2)} \right)^{2/3}.$$

(c) For finite Δ , we still have the implicit nonlinear equation to solve, but in the limit $\Delta \gg k_B T_c$, we can expand $T_c(\Delta) = T_c^\infty + \Delta T_c(\Delta)$. We may then set $T_c(n, \Delta)$ to $T_c^\infty(n)$ in the second term of our nonlinear implicit equation, move this term to the LHS, whence

$$\zeta(3/2) \lambda_{T_c}^{-3} \approx n - \text{Li}_{3/2}(e^{-2\Delta/k_B T_c^\infty}) \lambda_{T_c^\infty}^{-3}.$$

which is a simple algebraic equation for $T_c(n, \Delta)$. The second term on the RHS is tiny since $\Delta \gg k_B T_c^\infty$. We then find

$$T_c(n, \Delta) = T_c^\infty(n) \left\{ 1 - \frac{3}{2} e^{-2\Delta/k_B T_c^\infty(n)} + \mathcal{O}(e^{-4\Delta/k_B T_c^\infty(n)}) \right\}.$$