PHYS 201 Mathematical Physics, Fall 2017, Midterm

Due date: Tuesday, November 14th, 2017

Rules: Open book and without help from another person.

1. (10 pts) *Curvilinear transformations in the complex plane*: Consider a curvilinear transformation from the two-dimensional Cartesian coordinates, $(x, y) \to (u, v) \equiv (u(x, y), v(x, y))$. We may conversely write the inverse transformations $(x, y) = (x(u, v), y(u, v))$. We will assume a transformation such that the contours defined by $u(x, y) = \text{constant}$ and $v(x, y) = \text{constant}$ intersect at right angles for every x, y, i.e., the local basis vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ in the curvilinear system are orthogonal. Defining $h_1 \equiv \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}$ and $h_2 \equiv \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2}$, the local basis vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ of u and v respectively are given by

$$\boldsymbol{p} = \frac{1}{h_1}\frac{\partial x}{\partial u}\boldsymbol{i} + \frac{1}{h_1}\frac{\partial y}{\partial u}\boldsymbol{j},$$
$$\boldsymbol{q} = \frac{1}{h_2}\frac{\partial x}{\partial v}\boldsymbol{i} + \frac{1}{h_2}\frac{\partial y}{\partial v}\boldsymbol{j},$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the usual basis vectors in rectangular coordinates. Note that $\boldsymbol{p}.\boldsymbol{p} = \boldsymbol{q}.\boldsymbol{q} = 1$ and we have assumed u and v such that $\boldsymbol{p}.\boldsymbol{q} = 0$.

i. (1.5 pts) Show that the gradient operator in the curvilinear basis is given by $\tilde{\boldsymbol{\nabla}} = \left(\frac{1}{h_1}\frac{\partial}{\partial u}, \frac{1}{h_2}\frac{\partial}{\partial v}\right)$. Below we will use the Laplacian in this basis, $\tilde{\Delta}$, which can be derived from the divergence and gradient: (you don't have to show this)

$$\tilde{\Delta} = \frac{1}{h_1 h_2}\frac{\partial}{\partial u}\frac{h_2}{h_1}\frac{\partial}{\partial u} + \frac{1}{h_1 h_2}\frac{\partial}{\partial v}\frac{h_1}{h_2}\frac{\partial}{\partial v}.$$

(Hint: For the gradient, consider the projection of the usual gradient in rectangular coordinates along the $\boldsymbol{p}$ and $\boldsymbol{q}$ directions).

ii. (1.5 pts) Now, consider a conformal mapping $w = f(z)$ from the complex z-plane to the w-plane, i.e., $z = (x, y) \to w = (u, v)$. Show that, in this case, $h_1 = h_2 \equiv h$ and verify that $\boldsymbol{p}.\boldsymbol{q} = 0$. Further, using the form of the Laplacian from (i), show that a solution $\psi(z)$ of Laplace's equation $\Delta\psi = 0$ in the z-plane is also a solution of Laplace's equation $\tilde{\Delta}\psi = 0$ in the w-plane.

iii. (2 pts) Suppose $\phi(z)$ is a solution of the Helmholtz equation, $(\Delta + k^2)\,\phi = 0$, in the z-plane. Show that a sufficient condition for $\phi(w)$ to be separable solution of the Helmholtz equation in the w-plane, i.e., $\phi(w) = \phi_1(u)\phi_2(v)$ is that the scale factor h has the form $h^2 = g_1(u) + g_2(v)$, where g_1, g_2 are arbitrary functions.

iv. (3 pts) Show that the above separability condition for h is equivalent to the condition $\frac{\partial^2}{\partial u \partial v}\left(|\frac{dz}{dw}|^2\right) = 0$. Show that

$$\frac{\partial^2}{\partial u \partial v} = i\frac{\partial^2}{\partial w^2} - i\frac{\partial^2}{\partial \bar{w}^2},$$

and considering that dz/dw does not depend on $\bar{w}$ and vice-versa, obtain the separability conditions

$$\frac{d^2}{dw^2}\left(\frac{dz}{dw}\right) = \lambda\left(\frac{dz}{dw}\right); \quad \frac{d^2}{d\bar{w}^2}\left(\frac{d\bar{z}}{d\bar{w}}\right) = \lambda\left(\frac{d\bar{z}}{d\bar{w}}\right),$$

where λ is some constant.

v. (2 pts) Show that transformations of the form $z = \alpha + \beta w$ (which include rotations, changes of scale, and translation) and parabolic transformations $z = \frac{w^2}{2}$ satisfy the above equation for $\lambda = 0$. Finally, show that $z = e^w$ is a polar coordinate transformation (i.e., u = constant corresponds to circles of constant radius and v = constant corresponds to radial lines) and satisfies the separability condition for $\lambda = 1$.

2. (4 pts) Consider the Bromwich integral

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}F(s)ds,$$

where $t > 0$ and γ is a real number greater than the real part of all the singularities of F i.e., the path of integration is a vertical line to the right of all singularities. Show that $f(t)$ is also equal to the same integral over *any* other vertical line to the right of all singularities. Here, assume $|F(z)|$ grows slower than any exponential as $|z| \to \infty$ for z in the half plane where F is analytic.

3. (8 pts) Show that

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\exp\left[-b(z^2+a^2)^{1/2} + zt\right]}{(z^2+a^2)^{1/2}}dz = J_0\left(a\left(t^2 - b^2\right)^{1/2}\right),$$

where J_0 is the Bessel function of the first kind with integral representation

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\theta}d\theta.$$

Here a, b are real and > 0, and t is real with $t > b$. The path of integration is a vertical line to the right of all singularities. One possible method is to replace the integral over the vertical line to an integral around the branch cut joining ia and $-ia$. Explain why you can do this (you may refer to arguments made in Problem 2). To evaluate the integral around the cut, you would need to first consider b as purely imaginary, and then use analytic continuation to obtain the answer for real b.