PHYS 201 Mathematical Physics, Fall 2017, Homework 7

Due date: Thursday, December 7th, 2017

1. This exercise follows Arfken 10.5.3(a), 10.5.4 to 10.5.7.

   i. Show that the Green's function for the operator

   $$\mathcal{L}y(x) = \frac{d}{dx}\left(x\frac{dy(x)}{dx}\right)$$

   is

   $$G(x,t) = \begin{cases} -\ln t, & 0 \leq x < t \\ -\ln x, & t < x \leq 1 \end{cases} \tag{1}$$

   Due to the pathology at the boundary point, the solution to the inhomogeneous equation involving $\mathcal{L}$ in terms of the Green's function has an extra boundary term.

   ii. Show Green's theorem in one dimension for a Sturm-Liouville type operator:

   $$\int_a^b \left[u(t)\frac{d}{dt}\left(p(t)\frac{dv(t)}{dt}\right) - v(t)\frac{d}{dt}\left(p(t)\frac{du(t)}{dt}\right)\right]dt$$
   $$= \left[u(t)p(t)\frac{dv(t)}{dt} - v(t)p(t)\frac{du(t)}{dt}\right]\Bigg|_a^b$$

   iii. Using Green's theorem in the form above, let

   $$v(t) = y(t) \quad \text{and} \quad \frac{d}{dt}\left(p(t)\frac{dy(t)}{dt}\right) = -f(t)$$
   $$u(t) = G(x,t) \quad \text{and} \quad \frac{d}{dt}\left(p(t)\frac{\partial G(x,t)}{\partial t}\right) = -\delta(x-t)$$

   and show that it yields

   $$y(x) = \int_a^b G(x,t)f(t)dt + \left[G(x,t)p(t)\frac{dy(t)}{dt} - y(t)p(t)\frac{\partial}{\partial t}G(x,t)\right]\Bigg|_{t=a}^{t=b}$$

   iv. For $p(t) = t, y(t) = -t$ and $G(x,t)$ as in eq. 1, verify that the integrated part does not vanish.

2. Solve the following initial value problem for $t > 0$ in terms of Green's functions:

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = f(t), \quad y(0) = A, \frac{dy}{dt}(0) = B$$

3. Compute the inverse Laplace transform of

$$f(s) = \frac{k^2}{s(s^2 + k^2)}$$

i. By expanding in partial fractions, and

ii. From the calculus of residues.