PHYS 201 Mathematical Physics, Fall 2017, Homework 6

Due date: Thursday, November 30th, 2017

The first part of the homework is a review on some exact methods to solve ordinary differential equations. Read Chapter 1 of the book by Bender and Orszag for a more detailed version. For this section, you may turn in only the final answer for each problem; the derivation is not necessary. For the questions on approximate solutions, the full derivation is required.

Terminology: A linear(L) ODE is of the form $Ly(x) = f(x)$, where L is a differential operator

$$L = p_0(x) + p_1(x)\frac{d}{dx} + \cdots + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \frac{d^n}{dx^n}$$

If $f \equiv 0$, the differential equation is called homogeneous (H), otherwise it's an inhomogeneous (IH) differential equation. An n-th order homogeneous linear equation (H,L,n) has n linearly independent solutions $y_i(x), i = 1, 2, \ldots, n$. The Wronskian $W(x) = W[y_1(x), y_2(x), \ldots, y_n(x)]$ is used to test the linear independence of the solutions $\{y_i(x)\}$. Here

$$W(x) = \det \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If the Wronskian is non-zero except at isolated points, the solution set $\{y_i(x)\}$ is linearly independent. The Wronskian can also be found using Abel's formula, $W(x) = \exp\left[-\int^x p_{n-1}(t)dt\right]$.

1. *Constant-coefficient H,L,n ODEs:* If p_i's are constants, then substitute $y = e^{rx}$ and solve for the polynomial in r. If the root r is repeated m times, then the solutions are of the form $e^{rx}, xe^{rx}, \ldots x^{m-1}e^{rx}$.

   i. Solve $y''' - 4y'' + 5y' - 2y = 0$.

2. *Equidimensional-in-x:* If the equation is invariant under the transformation $x \to ax$, substitute $x = e^t$ and it turns into a constant-coefficient equation.

3. *Exact equations:* If a first-order equation can be written as

$$M[x, y(x)] + N[x, y(x)]y'(x) = \frac{d}{dx}f[x, f(x)] = 0$$

then the equation is exact (and the solution is $f = c$). A first-order equation is exact iff

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$

A higher order H,L equation can be converted to a lower order equation if we can observe that $Ly = 0 \Rightarrow \frac{d}{dx}(My) = 0$ where M is an L differential operator of one less order.

i. Use the integrating factor e^{xy} and verify that the equation $(1+xy+y^2)+(1+xy+x^2)y'(x) = 0$ is exact.

ii. Solve $yy'' + y'^2 - yy'/(1+x) = 0$.

4. *Reduction of order for L equations:* The order of any L equation can be reduced if one of the solution $y_1(x)$ is known by substituting $y(x) = u(x)y_1(x)$.

i. Solve $y'' + (x+2)y' + (1+x)y = 0$.

5. *Integrating factor for IH,L,1 equations:* All IH,L,1 equations can be solved using the integrating factor $I(x) = \exp[\int^x p_0(t)dt]$. The solution is

$$y(x) = \frac{c}{I(x)} + \frac{1}{I(x)} \int^x f(t)I(t)dt$$

i. Solve $y' + 2y = e^{3x}$.

6. *Variation of Parameters technique for IH,L,2 equations:* For every IH,L,2 equation $Ly = f$, if $y_1(x)$ and $y_2(x)$ are the solutions for the H,L,2 equation $Ly = 0$ and $W(x) = W[y_1(x), y_2(x)]$ is the Wronskian, then the solution $y(x)$ is given by

$$y(x) = -y_1(x) \int^x \frac{f(t)y_2(t)}{W(t)} dt + y_2(x) \int^x \frac{f(t)y_1(t)}{W(t)} dt$$

This technique can be generalized to higher order IH,L equations.

i. Solve $y'' + 6y + 9 = \cosh(x)$.

7. *Bernoulli equations:* Bernoulli equations are NL,1 equations and have the form

$$y' = a(x)y + b(x)y^P$$

They can be converted to L equations using the substitution $u(x) = y(x)^{1-P}$.

i. Solve $-xy' + y = xy^2, y(1) = 1$.

8. *Riccati equations:* These are NL,1 equations of the form

$$y'(x) = a(x)y^2(x) + b(x)y(x) + c(x)$$

They can be solved by guessing one particular solution $y_1(x)$ and substituting $y(x) = u(x)+y_1(x)$. This converts the Riccati equation to a Bernoulli equation and the general solution can be found thereafter.

i. Solve $xy' - 2y + ay^2 = bx^4$.

9. *Substitutions:* Sometimes a good substitution can convert an NL equation to an equation that is easily solvable.

i. Solve $y' = x^2 + 2xy + y^2$.

10. *Autonomous NL equations:* Autonomous equations are those in which the independent variable (in our case x) does not appear. The order of autonomous equations can be reduced to a lower order equation in y by using $y' = u(y), y'' = \frac{du}{dx} = u'(y)u(y)$ and so on.

i. Reduce the order and solve, if possible, the Blasius equation $y''' + yy'' = 0$.

11. *Scale-invariant NL equations:* If the transformations $x \to ax$ and $y \to a^P y$ leave the equation unchanged, then the equation is scale-invariant. The substitution $y(x) = x^P u(x)$ converts a scale-invariant equation to an equidimensional equation in x (which can be turned into an autonomous equation using $x = e^t$).

i. Reduce the order and solve, if possible, $xyy'' = yy' + xy'^2$.

12. *Equidimensional-in-y NL equations:* If the equation is unchanged by the transformation $y \to ay$, then the substitution $y(x) = e^{u(x)}$ reduces the order of the equation by one.

i. Reduce the order by one of $x^2yy'' + xy'y'' + yy' = 0$.

Approximate solutions to H,L ODEs:

13. Find the Frobenius series expansion about $x = 0$ of the solutions of the equation

$$2xy'' - y' + x^2y = 0$$

14. Find the full asymptotic behavior of the parabolic cylinder functions $D_\nu(x)$ as $x \to \infty$. The parabolic cylinder functions satisfy the differential equation

$$y'' + \left(\nu + \frac{1}{2} - \frac{x^2}{4}\right) y = 0$$

i. By substituting $y = e^{S(x)}$ in the differential equation and assuming $S'' \ll (S')^2, x \to \infty$, show that the two possible leading behaviors for the solution are

$$y(x) \sim c_1 x^{-\nu-1} e^{x^2/4}$$
$$y(x) \sim c_2 x^{\nu} e^{-x^2/4}$$

$D_\nu(x)$ is defined as the second equation with $c_2 = 1$. We have $D_\nu(x) = x^\nu e^{-x^2/4} w(x)$ where $w(x) = 1 + \epsilon(x), \epsilon(x) \ll 1$ as $x \to \infty$.

ii. Find the leading behavior of $\epsilon(x)$ of the form $a_1 x^{-\alpha}$ by substituting $D_\nu(x)$ back in the differential equation and solving the resulting differential equation in ϵ for large x (make appropriate approximations).

iii. Find the full asymptotic solution for $\epsilon(x)$ by substituting $\epsilon(x) = \sum_{n=1}^{\infty} a_n x^{-n\alpha}$ and solving the recurrence relation for a_n. Show that the series truncates if ν is a nonnegative integer (Observe that the parabolic cylinder equation for nonnegative integer ν is the Schrodinger's equation for the quantum harmonic oscillator. The solutions are known to be of the form $e^{-x^2/4}\mathrm{He}_\nu(x)$ i.e., the series solution found above generates the Hermite polynomials).

iv. For $\nu = 4.5$, calculate the number of terms in the optimal asymptotic approximation at $x = 0.5$ and $x = 5$.

15. Analyze the asymptotic behavior of the first Airy function $\mathrm{Ai}(x)$ as $x \to \infty$. The Airy equation is

$$y'' = xy$$

i. Find the two possible leading behaviors of the solutions using ideas from Exercise 14. $\mathrm{Ai}(x)$ is defined as the solution whose leading behavior is $y(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-2x^{3/2}/3}, x \to \infty$.

ii. Find the full asymptotic behavior i.e., the series solution to $w(x) = 1 + \epsilon(x)$ (as defined in 14) by directly substituting $w(x) = \sum_{n=0}^{\infty} a_n x^{-n\alpha}, \alpha > 0, a_0 = 1$ and solving for α and the obtained recurrence relation.

iii. Find the number of terms in the optimal asymptotic approximation for $x = 5$.