PHYS 201 Mathematical Physics, Fall 2017, Final

Due date: Thursday, December 14th, 2017.

Rules: Open book and without help from another person. Please contact the professor or TA if you have any questions.

1.

a. (5 pts) Show that

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{x-t}dt = e^{-x^2} + \frac{2i}{\sqrt{\pi}}Z(x), \text{ where } \quad Z(x) = e^{-x^2}\int_0^x e^{t^2}dt$$

where Re $x > 0$ and Im $x \neq 0$. You may use the integral representation

$$\frac{1}{x-t} = -2i\int_0^{\infty} e^{2i(x-t)k}dk.$$

b. (5 pts) If α and β are arbitrary complex numbers ($\alpha \neq \beta$), t is real and > 0, and if the path of integration is the vertical line Re $z = \gamma > 0$ to the right of all singularities, express

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(z+\alpha)^{1/2}e^{zt}}{z+\beta}dz$$

in terms of the function Z defined in part (a) and other terms. You may use the result in part (a) even if you do not show it.

2. The integral representation of the Airy function Ai(x) is given by

$$\text{Ai}(x) = \frac{1}{2\pi i}\int_C e^{xt-t^3/3}dt$$

where C is a contour which originates at $\infty e^{-2\pi i/3}$ and terminates at $\infty e^{2\pi i/3}$ (note that this integral can be related to $I(x^{3/2})$ in Homework 5, Problem 4 using the substitution $t = ix^{1/2}s$). The integral representation of the other Airy function Bi(x) is given by

$$\text{Bi}(x) = \frac{1}{2\pi}\int_{C_+} e^{xt-t^3/3}dt + \frac{1}{2\pi}\int_{C_-} e^{xt-t^3/3}dt,$$

where $C_\pm$ is a contour which originates at $\infty e^{\pm 2\pi i/3}$ and terminates at $+\infty$.

a. (5 pts) Show that, as $x \to -\infty$,

$$\text{Ai}(x) = \frac{1}{\sqrt{\pi}}(-x)^{-1/4}\sin\phi(x), \quad \phi(x) \sim \frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}.$$

(**Hint:** The steepest descent contour connecting $\infty e^{-2\pi i/3}$ to $\infty e^{2\pi i/3}$ can be deformed into two pieces, one passing through the saddle point at $t = -i\sqrt{-x}$ and one passing through the saddle point at $t = +i\sqrt{-x}$ (why?))

b. (5 pts) Using the method of steepest descents, find the asymptotic behavior of $\mathrm{Bi}(x)$ as $x \to +\infty$.

c. (5 pts) Find the asymptotic behavior of $\mathrm{Bi}(x)$ as $x \to -\infty$.

d. (5 pts) Using the above asymptotic relations, derive equation 10.5.4 from BO, i.e., the connection formula for the WKB approximation of the equation $\epsilon^2 y'' = Q(x)y$ across a turning point where Q vanishes linearly and has a *negative* slope.