Lecture 1: probability concepts I.

Bayesian probabilities in your non-academic life:

Example: The Monty Hall or
Let's Make a Deal Problem

- Three doors
- Car (prize) behind one door
- You pick a door, but don't open it yet
- Monty then opens one of the other doors, always revealing no car (he knows where it is)
- You now get to switch doors if you want
- Should you?
- Most people reason: Two remaining doors were equiprobable before, and nothing has changed. So doesn't matter whether you switch or not.

Laws of Probability

"There is this thing called *probability*. It obeys the laws of an axiomatic system. When identified with the real world, it gives (partial) information about the future."

- What axiomatic system?
- How to identify to real world?
  - Bayesian or frequentist viewpoints are somewhat different "mappings" from axiomatic probability theory to the real world
  - yet both are useful

"And, it gives a consistent and complete calculus of inference."

First, warmup exercise about frequentist notion of probabilities

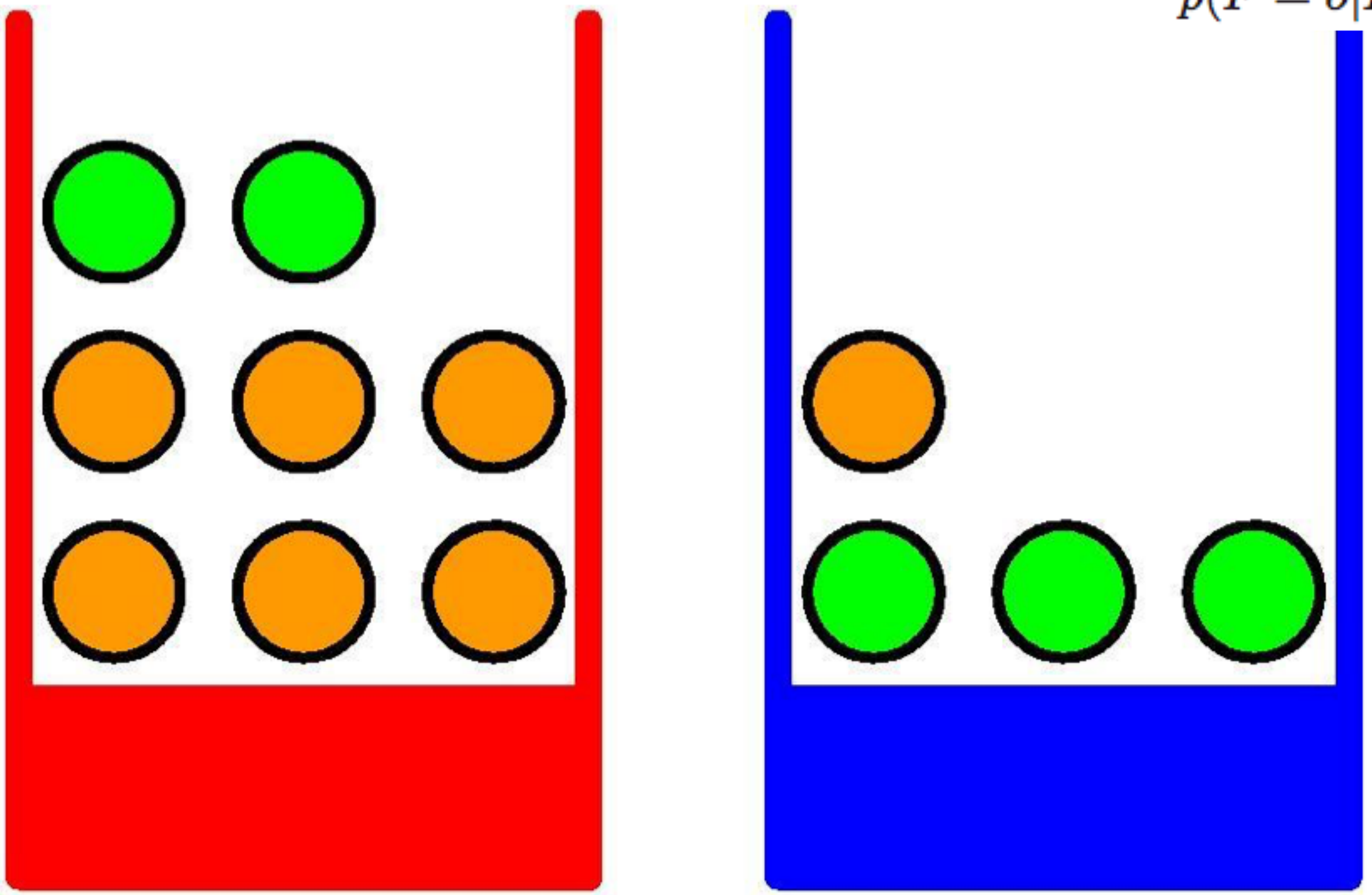
Probability Theory

joint probabilities
X and Y random variables

Apples and Oranges

$$p(B = r) = 4/10$$
$$p(B = b) = 6/10$$

$$p(F = a|B = r) = 1/4$$
$$p(F = o|B = r) = 3/4$$
$$p(F = a|B = b) = 3/4$$
$$p(F = o|B = b) = 1/4$$



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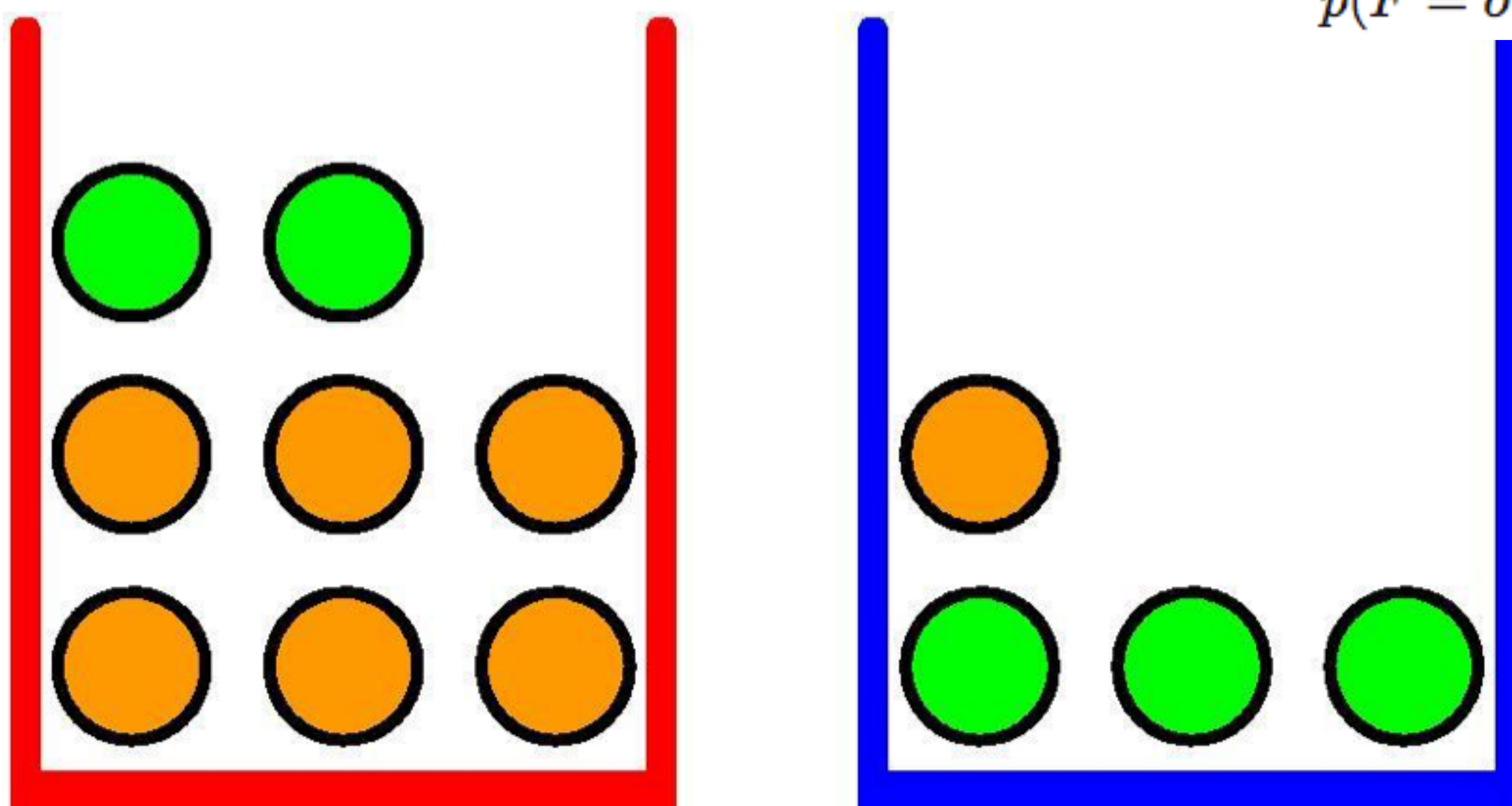
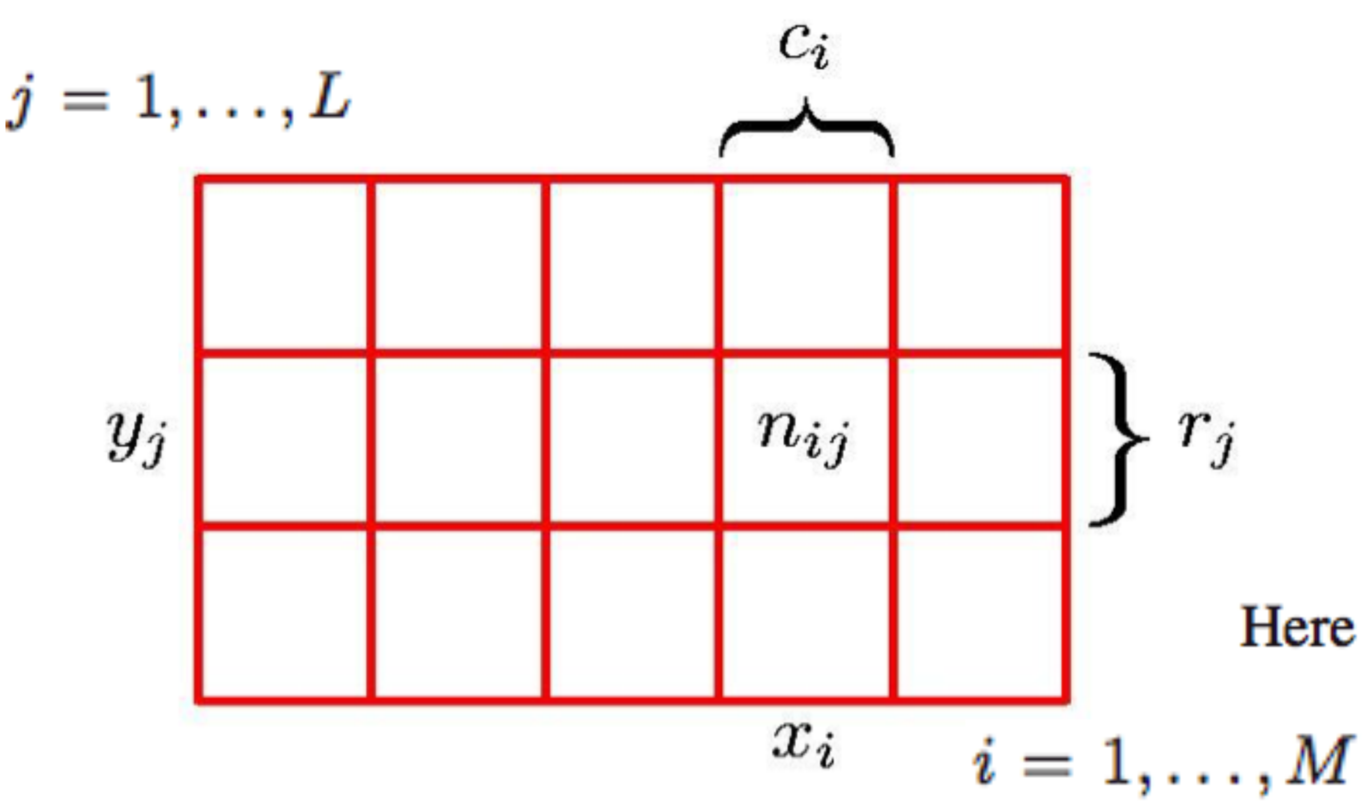


what is the probability to pick apple?
if orange, what is the probability that it came from blue box?

two elementary rules in probability theory help: sum rule and product rule

Probability Theory

joint probabilities
X and Y random variables



Marginal Probability

$$p(X = x_i) = \frac{c_i}{N}.$$

Here we are implicitly considering the limit $N \to \infty$

Joint Probability

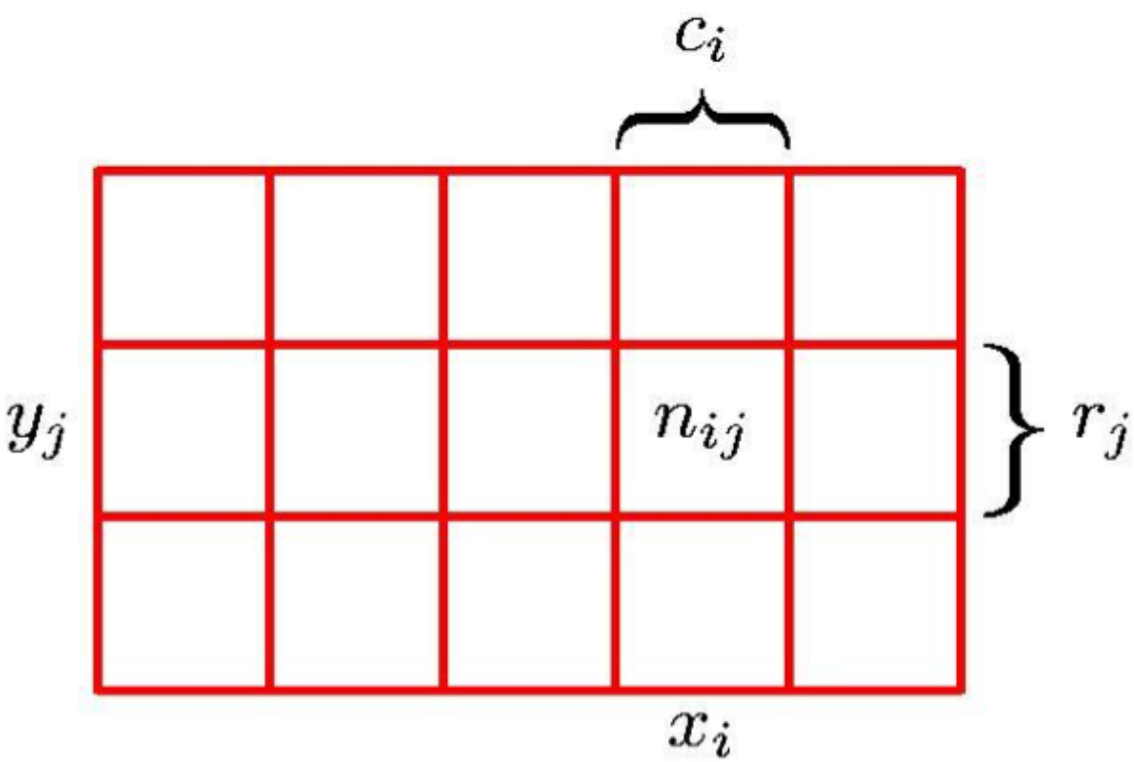
$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

Conditional Probability

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$

Probability Theory

joint probabilities
X and Y random variables



Sum Rule

$$p(X = x_i) = \frac{c_i}{N} = \frac{1}{N} \sum_{j=1}^{L} n_{ij}$$
$$= \sum_{j=1}^{L} p(X = x_i, Y = y_j)$$

Product Rule

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N}$$
$$= p(Y = y_j | X = x_i) p(X = x_i)$$

The Rules of Probability

joint probabilities

X and Y random variables

Sum Rule

$$p(X) = \sum_Y p(X, Y)$$

Product Rule

$$p(X, Y) = p(Y|X)p(X)$$

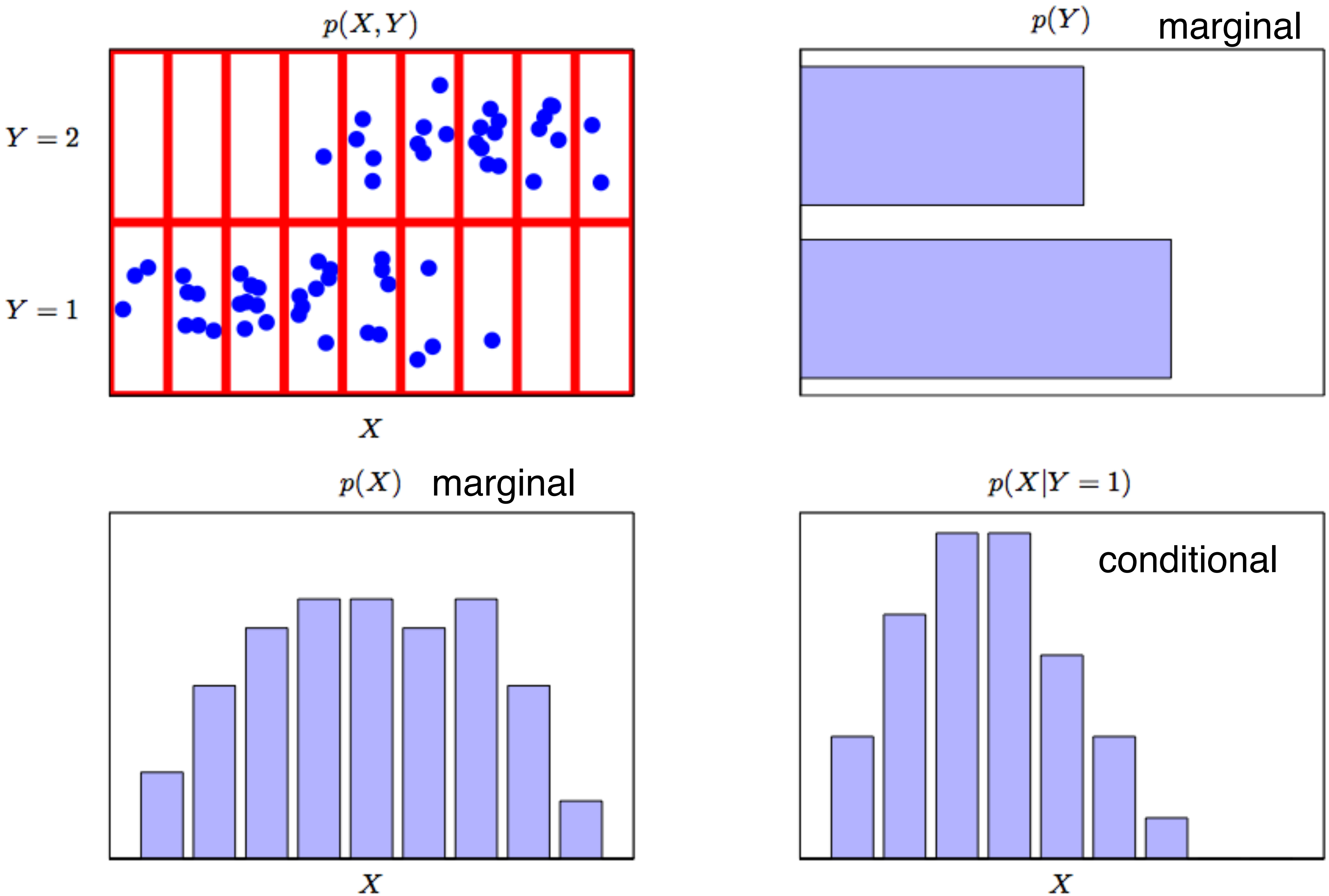
Bayes' Theorem

$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}$$

$$p(X) = \sum_{Y} p(X|Y)p(Y)$$ normalization

posterior $\propto$ likelihood × prior

tool: histogram of 60 events — joint probability distribution



return to the problem of two boxes with fruits

$$\begin{aligned} p(B=r) &= 4/10 \\ p(B=b) &= 6/10 \end{aligned}$$

marginal

$$p(B=r)+p(B=b)=1$$ normalization

$$\begin{aligned} p(F=a|B=r) &= 1/4 \\ p(F=o|B=r) &= 3/4 \\ p(F=a|B=b) &= 3/4 \\ p(F=o|B=b) &= 1/4 \end{aligned}$$

conditional

$$p(F=a|B=r)+p(F=o|B=r)=1$$

normalization

$$p(F=a|B=b)+p(F=o|B=b)=1$$

$$\begin{aligned} p(F=a) &= p(F=a|B=r)p(B=r)+p(F=a|B=b)p(B=b) \\ &= \frac{1}{4}\times\frac{4}{10}+\frac{3}{4}\times\frac{6}{10}=\frac{11}{20} \end{aligned}$$

picking apple

$$p(F=o)=1-11/20=9/20$$

picking orange

return to the problem of two boxes with fruits

if orange was picked, what was the probability of the box color ?

using Bayes' theorem, we can reverse the conditional probabilities:

$$p(B=r|F=o) = \frac{p(F=o|B=r)p(B=r)}{p(F=o)} = \frac{3}{4} \times \frac{4}{10} \times \frac{20}{9} = \frac{2}{3}$$

and from the sum rule:

$$p(B=b|F=o) = 1 - 2/3 = 1/3$$

return to the problem of two boxes with fruits

if orange was picked, what was the probability of the box ?

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interpretation of Bayes' theorem:
p(B) *prior probability*, if we are told that blue box was chosen available before we observe the fruit

Once we are told it was orange, we can use Bayes' theorem to calculate p(BIF) which is the *posterior probability*