

6-24 After rearrangement, the Schrödinger equation is $\frac{d^2\psi}{dx^2} = \left(\frac{2m}{\hbar^2}\right)\{U(x) - E\}\psi(x)$ with $U(x) = \frac{1}{2}m\omega^2 x^2$ for the quantum oscillator. Differentiating $\psi(x) = Cxe^{-\alpha x^2}$ gives

$$\frac{d\psi}{dx} = -2\alpha x\psi(x) + C^{-\alpha x^2}$$

and

$$\frac{d^2\psi}{dx^2} = -\frac{2\alpha x d\psi}{dx} - 2\alpha\psi(x) - (2\alpha x)Ce^{-\alpha x^2} = (2\alpha x)^2\psi(x) - 6\alpha\psi(x).$$

Therefore, for $\psi(x)$ to be a solution requires $(2\alpha x)^2 - 6\alpha = \frac{2m}{\hbar^2}\{U(x) - E\} = \left(\frac{m\omega}{\hbar}\right)^2 x^2 - \frac{2mE}{\hbar^2}$.

Equating coefficients of like terms gives $2\alpha = \frac{m\omega}{\hbar}$ and $6\alpha = \frac{2mE}{\hbar^2}$. Thus, $\alpha = \frac{m\omega}{2\hbar}$ and

$E = \frac{3\alpha\hbar^2}{m} = \frac{3}{2}\hbar\omega$. The normalization integral is $1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2C^2 \int x^2 e^{-2\alpha x^2} dx$ where the second step follows from the symmetry of the integrand about $x = 0$. Identifying a with 2α in the integral of Problem 6-32 gives $1 = 2C^2 \left(\frac{1}{8\alpha}\right) \left(\frac{\pi}{2\alpha}\right)^{1/2}$ or $C = \left(\frac{32\alpha^3}{\pi}\right)^{1/4}$.

6-25 At its limits of vibration $x = \pm A$ the classical oscillator has all its energy in potential form:

$E = \frac{1}{2}m\omega^2 A^2$ or $A = \left(\frac{2E}{m\omega^2}\right)^{1/2}$. If the energy is quantized as $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, then the

corresponding amplitudes are $A_n = \left[\frac{(2n+1)\hbar}{m\omega}\right]^{1/2}$.

6-32 The probability density for this case is $|\psi_0(x)|^2 = C_0^2 e^{-ax^2}$ with $C_0 = \left(\frac{a}{\pi}\right)^{1/4}$ and $a = \frac{m\omega}{\hbar}$.

For the calculation of the average position $\langle x \rangle = \int_{-\infty}^{\infty} x |\psi_0(x)|^2 dx$ we note that the integrand is an odd function, so that the integral over the negative half-axis $x < 0$ exactly cancels that over the positive half-axis ($x > 0$), leaving $\langle x \rangle = 0$. For the calculation of $\langle x^2 \rangle$, however, the integrand $x^2 |\psi_0|^2$ is symmetric, and the two half-axes contribute equally, giving

$$\langle x^2 \rangle = 2C_0^2 \int_0^{\infty} x^2 e^{-ax^2} dx = 2C_0^2 \left(\frac{1}{4a}\right) \left(\frac{\pi}{a}\right)^{1/2}.$$

Substituting for C_0 and a gives $\langle x^2 \rangle = \frac{1}{2a} = \frac{\hbar}{2m\omega}$ and $\Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$.

6-33 (a) Since there is no preference for motion in the leftward sense vs. the rightward sense, a particle would spend equal time moving left as moving right, suggesting $\langle p_x \rangle = 0$.

(b) To find $\langle p_x^2 \rangle$ we express the average energy as the sum of its kinetic and potential energy contributions: $\langle E \rangle = \langle \frac{p_x^2}{2m} \rangle + \langle U \rangle = \frac{\langle p_x^2 \rangle}{2m} + \langle U \rangle$. But energy is sharp in the oscillator ground state, so that $\langle E \rangle = E_0 = \frac{1}{2} \hbar \omega$. Furthermore, remembering that $U(x) = \frac{1}{2} m \omega^2 x^2$ for the quantum oscillator, and using $\langle x^2 \rangle = \frac{\hbar}{2m\omega}$ from Problem 6-32, gives $\langle U \rangle = \frac{1}{2} m \omega^2 \langle x^2 \rangle = \frac{1}{4} \hbar \omega$. Then

$$\langle p_x^2 \rangle = 2m(E_0 - \langle U \rangle) = 2m\left(\frac{\hbar\omega}{4}\right) = \frac{m\hbar\omega}{2}.$$

(c)
$$\Delta p_x = \left(\langle p_x^2 \rangle - \langle p_x \rangle^2 \right)^{1/2} = \left(\frac{m\hbar\omega}{2} \right)^{1/2}$$

6-34 From Problems 6-32 and 6-33, we have $\Delta x = \left(\frac{\hbar}{2m\omega} \right)^{1/2}$ and $\Delta p_x = \left(\frac{m\hbar\omega}{2} \right)^{1/2}$. Thus,

$\Delta x \Delta p_x = \left(\frac{\hbar}{2m\omega} \right)^{1/2} \left(\frac{m\hbar\omega}{2} \right)^{1/2} = \frac{\hbar}{2}$ for the oscillator ground state. This is the minimum uncertainty product permitted by the uncertainty principle, and is realized only for the ground state of the quantum oscillator.

6-35 Applying the momentum operator $[p_x] = \left(\frac{\hbar}{i} \right) \frac{d}{dx}$ to each of the candidate functions yields

(a)
$$[p_x] \{ A \sin(kx) \} = \left(\frac{\hbar}{i} \right) k \{ A \cos(kx) \}$$

(b)
$$[p_x] \{ A \sin(kx) - A \cos(kx) \} = \left(\frac{\hbar}{i} \right) k \{ A \cos(kx) + A \sin(kx) \}$$

(c)
$$[p_x] \{ A \cos(kx) + iA \sin(kx) \} = \left(\frac{\hbar}{i} \right) k \{ -A \sin(kx) + iA \cos(kx) \}$$

(d)
$$[p_x] \{ e^{ik(x-a)} \} = \left(\frac{\hbar}{i} \right) ik \{ e^{ik(x-a)} \}$$

In case (c), the result is a multiple of the original function, since

$$-A \sin(kx) + iA \cos(kx) = i \{ A \cos(kx) + iA \sin(kx) \}.$$

The multiple is $\left(\frac{\hbar}{i}\right)(ik) = \hbar k$ and is the eigenvalue. Likewise for (d), the operation $[p_x]$ returns the original function with the multiplier $\hbar k$. Thus, (c) and (d) are eigenfunctions of $[p_x]$ with eigenvalue $\hbar k$, whereas (a) and (b) are not eigenfunctions of this operator.

7-1 (a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or $R = \frac{\left|\left(\frac{1}{2}\right)(1-i)\right|^2}{\left|\left(\frac{1}{2}\right)(1+i)\right|^2}$. But

$$|1-i|^2 = (1-i)(1-i)^* = (1-i)(1+i) = |1+i|^2 = 2, \text{ so that } R = 1 \text{ in this case.}$$

(b) To the left of the step the particle is free. The solutions to Schrödinger's equation are $e^{\pm ikx}$ with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$. To the right of the step $U(x) = U$ and the equation is $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(U-E)\psi(x)$. With $\psi(x) = e^{-kx}$, we find $\frac{d^2\psi}{dx^2} = k^2\psi(x)$, so that $k = \left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. Substituting $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$ shows that $\left[\frac{E}{(U-E)}\right]^{1/2} = 1$ or $\frac{E}{U} = \frac{1}{2}$.

(c) For 10 MeV protons, $E = 10 \text{ MeV}$ and $m = \frac{938.28 \text{ MeV}}{c^2}$. Using $\hbar = 197.3 \text{ MeV fm}/c$ ($1 \text{ fm} = 10^{-15} \text{ m}$), we find

$$\delta = \frac{1}{k} = \frac{\hbar}{(2mE)^{1/2}} = \frac{197.3 \text{ MeV fm}/c}{\left[(2)(938.28 \text{ MeV}/c^2)(10 \text{ MeV})\right]^{1/2}} = 1.44 \text{ fm}.$$

7-2 (a) To the left of the step the particle is free with kinetic energy E and corresponding wavenumber $k_1 = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad x \leq 0$$

To the right of the step the kinetic energy is reduced to $E - U$ and the wavenumber is now $k_2 = \left[\frac{2m(E-U)}{\hbar^2}\right]^{1/2}$

$$\psi(x) = Ce^{ik_2x} + De^{-ik_2x} \quad x \geq 0$$

with $D = 0$ for waves incident on the step from the left. At $x = 0$ both ψ and $\frac{d\psi}{dx}$ must be continuous: $\psi(0) = A + B = C$

$$\left.\frac{d\psi}{dx}\right|_0 = ik_1(A - B) = ik_2C.$$

(b) Eliminating C gives $A + B = \frac{k_1}{k_2}(A - B)$ or $A\left(\frac{k_1}{k_2} - 1\right) = B\left(\frac{k_1}{k_2} + 1\right)$. Thus,

$$R = \frac{|B|^2}{|A|^2} = \frac{(k_1/k_2 - 1)^2}{(k_1/k_2 + 1)^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

$$T = 1 - R = \frac{4k_1k_2}{(k_1 + k_2)^2}$$

(c) As $E \rightarrow U$, $k_2 \rightarrow 0$, and $R \rightarrow 1$, $T \rightarrow 0$ (no transmission), in agreement with the result for any energy $E < U$. For $E \rightarrow \infty$, $k_1 \rightarrow k_2$ and $R \rightarrow 0$, $T \rightarrow 1$ (perfect transmission) suggesting correctly that very energetic particles do not see the step and so are unaffected by it.

7-3 With $E = 25$ MeV and $U = 20$ MeV, the ratio of wavenumber is

$$\frac{k_1}{k_2} = \left(\frac{E}{E-U}\right)^{1/2} = \left(\frac{25}{25-20}\right)^{1/2} = \sqrt{5} = 2.236. \text{ Then from Problem 7-2 } R = \frac{(\sqrt{5}-1)^2}{(\sqrt{5}+1)^2} = 0.146$$

and $T = 1 - R = 0.854$. Thus, 14.6% of the incoming particles would be reflected and 85.4% would be transmitted. For electrons with the same energy, the transparency and reflectivity of the step are unchanged.

7-4 The reflection coefficient for this case is given in Problem 7-2 as

$$R = \frac{|B|^2}{|A|^2} = \frac{(k_1/k_2 - 1)^2}{(k_1/k_2 + 1)^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}.$$

The wavenumbers are those for electrons with kinetic energies $E = 54.0$ eV and $E - U = 54.0$ eV + 10.0 eV = 64.0 eV:

$$\frac{k_1}{k_2} = \left(\frac{E}{E-U}\right)^{1/2} = \left(\frac{54 \text{ eV}}{64 \text{ eV}}\right)^{1/2} = 0.9186.$$

Then, $R = \frac{(0.9186 - 1)^2}{(0.9186 + 1)^2} = 1.80 \times 10^{-3}$ is the fraction of the incident beam that is reflected at the boundary.

7-5 (a) The transmission probability according to Equation 7.9 is

$$\frac{1}{T(E)} = 1 + \left[\frac{U^2}{4E(U-E)} \right] \sinh^2 \alpha L \text{ with } \alpha = \frac{[2m(U-E)]^{1/2}}{\hbar}. \text{ For } E \ll U, \text{ we find}$$

$$(\alpha L)^2 \approx \frac{2mUL^2}{\hbar^2} \gg 1 \text{ by hypothesis. Thus, we may write } \sinh \alpha L \approx \frac{1}{2} e^{\alpha L}. \text{ Also}$$

$$U - E \approx U, \text{ giving } \frac{1}{T(E)} \approx 1 + \left(\frac{U}{16E}\right) e^{2\alpha L} \approx \left(\frac{U}{16E}\right) e^{2\alpha L} \text{ and a probability for}$$

$$\text{transmission } P = T(E) = \left(\frac{16E}{U}\right) e^{-2\alpha L}.$$

(b) Numerical Estimates: ($\hbar = 1.055 \times 10^{-34}$ Js)

- 1) For $m = 9.11 \times 10^{-31}$ kg, $U - E = 1.60 \times 10^{-21}$ J, $L = 10^{-10}$ m ;
 $\alpha = \frac{[2m(U-E)]^{1/2}}{\hbar} = 5.12 \times 10^8 \text{ m}^{-1}$ and $e^{-2\alpha L} = 0.90$
- 2) For $m = 9.11 \times 10^{-31}$ kg, $U - E = 1.60 \times 10^{-19}$ J, $L = 10^{-10}$ m ;
 $\alpha = 5.12 \times 10^9 \text{ m}^{-1}$ and $e^{-2\alpha L} = 0.36$
- 3) For $m = 6.7 \times 10^{-27}$ kg, $U - E = 1.60 \times 10^{-13}$ J, $L = 10^{-15}$ m ;
 $\alpha = 4.4 \times 10^{14} \text{ m}^{-1}$ and $e^{-2\alpha L} = 0.41$
- 4) For $m = 8$ kg, $U - E = 1$ J, $L = 0.02$ m ; $\alpha = 3.8 \times 10^{34} \text{ m}^{-1}$ and
 $e^{-2\alpha L} = e^{-1.5 \times 10^{33}} \approx 0$

7-16 Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3755.8 \text{ MeV}/c^2$, the first approximation to the decay length δ is

$$\delta \approx \frac{\hbar}{(2mU)^{1/2}} = \frac{197.3 \text{ MeV fm}/c}{[2(3755.8 \text{ MeV}/c^2)(30 \text{ MeV})]^{1/2}} = 0.4156 \text{ fm}.$$

This gives an effective width for the (infinite) well of $R + \delta = 9.4156 \text{ fm}$, and a ground

state energy $E_1 = \frac{\pi^2 (197.3 \text{ MeV fm}/c)^2}{2(3755.8 \text{ MeV}/c^2)(9.4156 \text{ fm})^2} = 0.577 \text{ MeV}$. From this E we calculate

$U - E = 29.42 \text{ MeV}$ and a new decay length

$$\delta = \frac{197.3 \text{ MeV fm}/c}{[2(3755.8 \text{ MeV}/c^2)(29.42 \text{ MeV})]^{1/2}} = 0.4197 \text{ fm}.$$

This, in turn, increases the effective well width to 9.4197 fm and lowers the ground state energy to $E_1 = 0.576 \text{ MeV}$. Since our estimate for E has changed by only 0.001 MeV , we may be content with this value. With a kinetic energy of E_1 , the alpha particle in the

ground state has speed $v_1 = \left(\frac{2E_1}{m}\right)^{1/2} = \left[\frac{2(0.576 \text{ MeV})}{(3755.8 \text{ MeV}/c^2)}\right]^{1/2} = 0.0175c$. In order to be

ejected with a kinetic energy of 4.05 MeV , the alpha particle must have been preformed in an excited state of the nuclear well, not the ground state.

7-17 The collision frequency f is the reciprocal of the transit time for the alpha particle crossing the nucleus, or $f = \frac{v}{2R}$, where v is the speed of the alpha. Now v is found from the kinetic energy which, inside the nucleus, is not the total energy E but the difference $E - U$ between the total energy and the potential energy representing the bottom of the nuclear well. At the nuclear radius $R = 9 \text{ fm}$, the Coulomb energy is

$$\frac{k(Ze)(2e)}{R} = 2Z\left(\frac{ke^2}{a_0}\right)\left(\frac{a_0}{R}\right) = 2(88)(27.2 \text{ eV})\left(\frac{5.29 \times 10^4 \text{ fm}}{9 \text{ fm}}\right) = 28.14 \text{ MeV}.$$

From this we conclude that $U = -1.86 \text{ MeV}$ to give a nuclear barrier of 30 MeV overall. Thus an alpha with $E = 4.05 \text{ MeV}$ has kinetic energy $4.05 + 1.86 = 5.91 \text{ MeV}$ inside the nucleus. Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3755.8 \text{ MeV}/c^2$ this kinetic energy represents a speed

$$v = \left(\frac{2E_k}{m} \right)^{1/2} = \left[\frac{2(5.91)}{3755.8 \text{ MeV}/c^2} \right]^{1/2} = 0.056c.$$

Thus, we find for the collision frequency $f = \frac{v}{2R} = \frac{0.056c}{2(9 \text{ fm})} = 9.35 \times 10^{20} \text{ Hz}$.