→ Landau & Lifshitz, E. C. M.

→ Y. R. Shen, Prin. of NL Optics.

Application → **Self-Focusing**

Consider a beam propagating in a medium with

$$\frac{n}{n_0} = 1 + \frac{\Delta n}{n_0} |E|^2$$

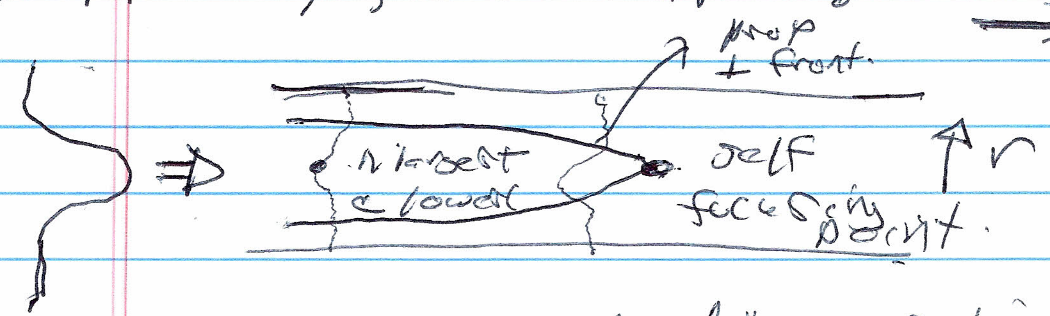
Kerr liquid

$\downarrow$ intensity dependent refractive index → nonlinear

$\boxed{\text{Theme: insight from mechanics}}$

→ qualitatively, what happens?

→ z



↓ beam profile (initial scale)

medium → high intensity ⇒ high n ⇒ low speed.

so beam self focuses

How describe?

$$\nabla^2 E + n^2 \frac{\omega^2}{c_0^2} E = 0$$

$$E = e^{ikz} \psi(z, r) \qquad k^2 = \frac{n_0^2 \omega^2}{c_0^2}$$

$\Rightarrow$

$$\cancel{-k_z^2\Psi} + 2ik_z\frac{\partial\Psi}{\partial z} + \cancel{h.o.t.} + \nabla_\perp^2\Psi$$

$$+ \cancel{n_0^2\frac{\omega^2}{c_0^2}\Psi} + n_2|\Psi|^2\frac{\omega^2}{c_0^2}\Psi = 0$$

$\Rightarrow$ so

$$\boxed{2ik_z\frac{\partial\Psi}{\partial z} + \nabla_\perp^2\Psi + n_2\frac{\omega^2}{c_0^2}|\Psi|^2\Psi = 0}$$

- parabolic wave eqn, but nonlinear
- NLS (i.e. Hartree - self consistent field)

Now, proceed as in HW.

$$\begin{cases} 2ik_z\dfrac{\partial\Psi}{\partial z} + \nabla_\perp\Psi = -k^2\dfrac{\Delta n}{n_0}\Psi \\[2ex] \Delta n/n_0 = n_2|\Psi|^2/n_0 \end{cases}$$

$$\psi = A e^{i\phi}$$, old eikonal

$\Rightarrow$

$$2ik_z \frac{\partial \psi}{\partial z} = 2ik_z \left(\frac{\partial A}{\partial z} e^{i\phi} + A i \frac{\partial \phi}{\partial z} \right)$$

$$\nabla_\perp^2 \psi = \nabla_\perp^2 \left(A e^{i\phi} \right) = \nabla_\perp \left(A i \nabla_\perp \phi \, e^{i\phi} + \nabla_\perp A \, e^{i\phi} \right)$$

$$= \left(A i \nabla_\perp^2 \phi + i 2 \nabla_\perp \phi \cdot \nabla_\perp A \, e^{i\phi} - A (\nabla_\perp \phi)^2 + \nabla_\perp^2 A \right) e^{i\phi}$$

Real $\rightarrow$ phase
im $\Rightarrow$ ampl.

$$+ (2ik_z)(i) A \frac{\partial \phi}{\partial z} e^{i\phi} - A (\nabla_\perp \phi)^2 e^{i\phi}$$

$$+ \nabla_\perp^2 A e^{i\phi} + k^2 \frac{\Delta n}{n_0} A e^{i\phi}$$

$\Rightarrow$

$$\frac{\partial \phi}{\partial z} + \frac{1}{2k} (\nabla_\perp \phi)^2 - \frac{k}{2} \left(\frac{\nabla_\perp^2 A}{k^2 A} + 2 \frac{\Delta n}{n} \right) = 0$$

so

$$\frac{\partial\phi}{\partial z} + \frac{1}{2k}(\nabla_\perp\phi)^2 - \frac{k}{2}\left(\frac{\nabla_\perp^2 A}{k^2 A} + \frac{2\Delta n}{n}\right) = 0$$

phase

$$k\frac{\partial}{\partial z}A^2 = -\nabla_\perp\cdot(A^2\nabla_\perp\phi)$$

amplitude

Now, given paraxial wave eqn $\leftrightarrow$ Schroedinger eqn.

no surprise that phase equation resembles H-J eqn.

i.e.

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V = 0$$

thus, can make analogy with particle dynamics, i.e.

$$V = -\frac{k}{2}\left(\frac{\nabla_\perp^2 A}{k^2 A} + \frac{2\Delta n}{n}\right)$$

For dynamics:

$z \leftrightarrow t$

$r \leftrightarrow z$

so

$$m \frac{d^2 z}{dt^2} = -\frac{\partial V}{\partial z}$$

$\Rightarrow$

$$\boxed{\frac{d^2 r}{dz^2} = -\frac{1}{k}\frac{\partial V}{\partial r}}$$

ray path of radius r vs z (k-scale)

i.e.

$$\boxed{\frac{d^2 r}{dz^2} = +\frac{\partial}{\partial r}\left[\frac{1}{2}\left(\frac{v_\perp^2 A}{k^2 A} + \frac{2\Delta n}{n}\right)\right]}$$

dynamical problem.

Now – formally particle dynamics, cn flavor.

– but need A

– A from coupled eqns. ϕ, A; involving Δn.

To make progress try approximate form for $A(r,z)$:

i.e.

$$A(r,z) = A_0 \frac{a_0^2}{a^2(z)} \exp\left[\frac{-r^2}{2a^2(z)}\right]$$

- paraxial approx.

- each ray follows $r/a \sim r_0/a_0$
$\Downarrow$
i.c.

so plugging in:

$$V = -k\left(\frac{-1}{k^2a^2} + \frac{r^2}{2k^2a^4} + \frac{\Delta n}{n_0}\right)$$

Now, for paraxial $\rightarrow$ most beam on axis

so $r^2 \ll a^2$

$$\Rightarrow \boxed{V(a) = -k\left(\frac{-1}{k^2a^2} + \frac{\Delta n}{n_0}\right)}$$

so, $$V(a) = -k\left(\frac{-1}{k^2 a^2} + \frac{\Delta n}{n_0}\right)$$

and
$$\frac{d^2 r}{dz^2} = -k^{-1}\frac{\partial V}{\partial r}$$
becomes

$$k\frac{d^2 a(z)}{dz^2} = -\frac{\partial V}{\partial a}$$

$*\ a' = da/dz$

$$k\frac{da}{dz}\frac{d^2 a}{dz^2} + \frac{da}{dz}\frac{\partial V}{\partial a}$$

$$\frac{d}{dz}\left\{\frac{k}{2}\left(\frac{da}{dz}\right)^2 + V(a)\right\} = 0$$

so at last:

$$\boxed{\underbrace{\frac{k}{2}\left(\frac{da}{dz}\right)^2}_{\text{K.E.}} + \underbrace{V(a)}_{\text{potential}} = \text{const.}}$$

energy conservation relation

self-focus $\Rightarrow V < 0$
$\rightarrow$ pulls in ray

$$\frac{k}{2}\left(\frac{da}{dz}\right)_0^2 + V(a_0) = \frac{k}{2}\left(\frac{da}{dz}\right)^2 + V(a)$$

$\Rightarrow$

$$z_f = \int_{a_0}^{a_{min}} \left\{ \frac{2}{k}\big(V(a_0) - V(a)\big) + \left(\frac{da}{dz}\right)_0^2 \right\}^{-1/2} da$$

z_f: focal length

Key competition: $\oplus$ $\ominus$

$$\boxed{V = -k\left[\underbrace{-\frac{1}{k^2 a^2}}_{\text{diffraction (repulsive pot.)}} + \underbrace{n_2 \frac{A_0^2}{n_0}}_{\text{(attr. potential)}} \right]}$$

so

$$V = \frac{1}{k a^2}\left(1 - \frac{P}{P_c}\right)$$

self-focusing for $P > P_c$!

$$P = n_0 c a^2 A_0^2 / \lambda \rightarrow \text{beam power}$$

$$P_c = c \lambda^2 / 8\pi n_2$$

crank

$\Rightarrow$

$$z_f = \frac{k a_0^2/\sqrt{2}}{\left(P/P_0 - 1\right)^{1/2} - \left(\frac{k a_0}{\sqrt{2}}\right)\left(da/dz\right)_0}$$

focal length.

Now, for exact result: [L-L / Section 109
Vlasov, Petrishchev,
Talanov, 1971

$$ik\frac{\partial}{\partial z}\psi + \frac{1}{2}\nabla_\perp^2\psi + \eta|\psi|^2\psi = 0$$

$\Rightarrow$ prob. cons.

analogue of prob. current

$$\frac{\partial}{\partial z}|\psi|^2 + \underline{\nabla}\cdot\underline{J} = 0$$

$$\underline{J} = \frac{i}{2k_0}\left[\psi\,\underline{\nabla}_\perp\psi^* - \psi^*\,\underline{\nabla}_\perp\psi\right]$$

Can show $\underline{\underline{2}}$ conserved quantities:

$$N = \int d^2r\,|\psi|^2 \quad \rightarrow \#, \text{ probability}$$

$$\mathcal{E} = \frac{1}{2k^2}\int d^2r \left\{ |\nabla_\perp \psi|^2 - \eta\left(|\psi|^2\right)^2 \right\}$$

energy

(direct)

Now, can define mean beam radius:

$$R^2(z) = \frac{1}{N}\int d^2r\, r^2 |\psi|^2 \qquad \langle r^2 \rangle$$

and $\frac{d}{dz} R^2(z) \Rightarrow$

$$\boxed{R^2(z) = 2\mathcal{E}(z-z_0)^2/N + R_0^2}$$

$\Rightarrow \mathcal{E} < 0 \Rightarrow$ collapse to focal point.