Quantum Mechanics PHYS 212B

Problem Set 7

Due Thursday, March 3, 2016

Exercise 7.1 N identical spin-1/2 otherwise non-interacting particles move in a collective one-dimensional harmonic oscillator potential.

(a) What is the ground state energy?

(b) What if we assume that N is very large and the individual single particle wave functions can be approximated as plane waves?

(c) What happens if we do the latter limit in a three-dimensional harmonic oscillator potential?

Solution 7.1

(a) With fermions, each level can only allow two particles. Allocate the particles from the ground level $\frac{1}{2}\hbar\omega$, up to $(\frac{1}{2}+[N/2])\hbar\omega$, the total energy of ground state is

$$E_{\text{total}} = \begin{cases} \frac{N^2}{4}\hbar\omega & \text{if } N \text{ is even} \\ \frac{N^2+1}{4}\hbar\omega & \text{if } N \text{ is odd} \end{cases}$$

(b) In one-dimensional case, the degeneracy across each energy level is uniform, we can assume the density as function of engery

$$\rho(E) = \gamma$$

where γ is a constant. For N particles,

$$N = \int_0^{E_f} \rho(E)dE = \gamma E_f$$

then we have

$$\gamma = \frac{N}{E_f} = \frac{N}{\frac{N}{2}\hbar\omega} = \frac{2}{\hbar\omega}$$

The total energy

$$E_{\text{total}} = \int_0^{E_f} E\rho(E)dE = \frac{1}{2}\gamma E_f^2 = \frac{N^2}{4}\hbar\omega$$

(c) In three-dimensional case, the energy level is

$$E_n = (n_x + n_y + n_z + \frac{1}{2})\hbar\omega$$

The degeneracy for $n = n_x + n_y + n_z$ is $\frac{1}{2}(n+1)(n+2)$, and we have

$$N = 2\sum_{n=0}^{n_f} \frac{1}{2}(n+1)(n+2) = \frac{1}{3}n_f(n_f+1)(n_f+3) \approx \frac{1}{3}n_f^3$$

so that

$$n_f = (3N)^{1/3}$$

The total energy

$$E_{\text{total}} = 2\sum_{n=0}^{n_f} \frac{1}{2}(n+1)(n+2)(n+\frac{1}{2})\hbar\omega \approx \frac{1}{4}n_f^4\hbar\omega = \frac{(3N)^{4/3}}{4}\hbar\omega$$

Knowing the degeneracy for $n = n_x + n_y + n_z$ is $\frac{1}{2}(n+1)(n+2)$, so that the state density as a function of energy is

$$\rho(E) = \gamma E^2$$

With N particles,

$$N = \int_0^{E_f} \rho(E)dE = \frac{1}{3}\gamma E_f^3$$

so that

$$\gamma = \frac{3N}{E_f^3}$$

The total energy

$$E_{\text{total}} = \int_0^{E_f} \rho(E)EdE = \frac{1}{4}\gamma E_f^4 = \frac{3}{4}NE_f = \frac{(3N)^{4/3}}{4}\hbar\omega$$

Exercise 7.2 Two identical spin-1/2 particles move in one dimension under the influence of the infinite-wall potential $V = \infty$ for $x < 0, x > L$, and $V = 0$ for $0 \le x \le L$.

(a) What is the ground state wave function and the ground state energy when the two particles are constrained to be in the spin-triplet state?

(b) Same question as **(a)** but now the particles are in the spin-singlet state?

(c) Now suppose that the particles interact mutually via a very short range attractive potential, $V = -\lambda\delta(x_1 - x_2)$ (with $\lambda > 0$). Use perturbation theory for this mutual potential and discuss what happens to the energy levels in parts **(a)** and **(b)** above.

Solution 7.2 The wave function and its energy level are

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \qquad E_n = \frac{\pi^2\hbar^2}{8ma^2}n^2$$

(a) For the spin-triplet state, the spatial wave function needs to be antisymmetric

$$\psi(x_1, x_2) = [\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)]$$

The ground state energy

$$E_{\text{ground}} = \frac{\pi^2\hbar^2}{8ma^2}(1^2 + 2^2) = \frac{5\pi^2\hbar^2}{8ma^2}$$

(b) For the spin-singlet state, the spatial wave function needs to be symmetric,

$$\psi(x_1, x_2) = \psi_1(x_1)\psi_1(x_2)$$

The ground state energy

$$E_{\text{ground}} = \frac{\pi^2\hbar^2}{8ma^2}(1^2 + 1^2) = \frac{2\pi^2\hbar^2}{8ma^2}$$

(c) For the spin-triplet state, $\psi(x_1, x_2)$ is antisymmetric, so

$$\Delta E = \int_0^L dx_1 \int_0^L dx_2 \psi(x_1, x_2)\psi(x_1, x_2)[-\lambda\delta(x_1 - x_2)] = 0.$$

For the spin-singlet state,

$$\begin{aligned}\Delta E &= \int_0^L dx_1 \int_0^L dx_2 \psi(x_1, x_2)\psi(x_1, x_2)[-\lambda\delta(x_1 - x_2)] \\ &= -\lambda \int_0^L dx \frac{4}{L^2} \sin^4\left(\frac{\pi x}{L}\right) \\ &= -\frac{3\lambda}{2L}\end{aligned}$$