Quantum Mechanics PHYS 212B

Problem Set 2

Due Thursday, January 21, 2016

Exercise 2.1 Consider a two state system governed by Hamiltonian H and with energy eigenstates $|E_1\rangle$ and $|E_2\rangle$ where $H|E_1\rangle = E_1|E_1\rangle$ and $H|E_2\rangle = E_2|E_1\rangle$. Two other states are:

$$\begin{aligned}|x\rangle =& \frac{|E_1\rangle + |E_2\rangle}{\sqrt{2}} \\ |y\rangle =& \frac{|E_1\rangle - |E_2\rangle}{\sqrt{2}}\end{aligned}$$

At time $t = 0$ the system is in state $|x\rangle$. At what subsequent times is the probability to find the system in state $|y\rangle$ biggest and what is this probability?

Solution 2.1 Time evolution operator

$$u(t, t_0) = e^{-iH(t-t_0)/\hbar}$$

The state of the system $|\psi(t)\rangle$ with $|\psi(t = 0)\rangle = |x\rangle$,

$$|\psi(t)\rangle = e^{-iHt/\hbar}|x\rangle$$

Then

$$\begin{aligned}\langle y|\psi(t)\rangle &= \frac{1}{2}(\langle E_1| - \langle E_2|)e^{-iHt/\hbar}(|E_1\rangle + |E_2\rangle) \\ &= \frac{1}{2}\left(e^{-iE_1 t/\hbar} - e^{-iE_2 t/\hbar}\right)\end{aligned}$$

The probability to find state $|y\rangle$,

$$P(|y\rangle) = |\langle y|\psi(t)\rangle|^2 = \frac{1}{2}\left[1 - \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\right]$$

The biggest probability

$$P_{\max}(|y\rangle) = 1$$

when

$$t = \frac{(2n+1)\pi\hbar}{E_1 - E_2}.$$

Exercise 2.2 The usual Hydrogen-like wave functions for single atoms are derived with the assumption that finite size of the nucleus can be neglected. Model the nucleus as an uniformly charged spherical shell of radius R_0. What is the first order shift in energy of the $2s$ state? Assume that $R_0 \ll a_0$, where $a_0 = 4\pi\epsilon_0(\hbar c)^2/(m_e c^2 Z^2 e^2)$ is the appropriate Bohr radius for a single electron atom with nuclear charge Z. The spatial wave function of the unperturbed state is

$$u(r) = (32\pi)^{-1/2}\left(\frac{Z}{a_0}\right)^{3/2}\left(2 - \frac{Zr}{a_0}\right)e^{-Zr/2a_0}.$$

Solution 2.2 The perturbation is

$$H' = \frac{Ze^2}{4\pi\epsilon_0 r}, \quad 0 < r < R_0$$

Then the first order energy shift is

$$\begin{aligned}
\langle 2s|H'|2s\rangle &= 4\pi \int_0^{R_0} u(r)H'u(r)r^2 dr \\
&= 4\pi \int_0^{R_0} \frac{1}{32\pi}\left(\frac{Z}{a_0}\right)^3 \left(2 - \frac{Zr}{a_0}\right)^2 e^{-\frac{Zr}{a_0}} \frac{Ze^2}{4\pi\epsilon_0 r} r^2 dr
\end{aligned}$$

Let $r' = Zr/a_0$,

$$\begin{aligned}
\langle 2s|H'|2s\rangle &= \frac{Z^2e^2}{32\pi\epsilon_0 a_0} \int_0^{\frac{ZR_0}{a_0}} (2 - r')^2 r' e^{-r'} dr' \\
&\approx \frac{Z^2e^2}{32\pi\epsilon_0 a_0} \int_0^{\frac{ZR_0}{a_0}} (2 - r')^2 r'(1 - r') dr' \\
&\approx \frac{Z^2e^2}{32\pi\epsilon_0 a_0} \int_0^{\frac{ZR_0}{a_0}} 4r' dr' \\
&= \frac{Z^4 e^2 R_0^2}{16\pi\epsilon_0 a_0^3}
\end{aligned}$$

since $R_0 \ll a_0$, i.e. $r' \ll 1$, $e^{-r'} \approx 1 - r'$ and only the first-order terms of r' are kept in the integral kernel.