Quantum Mechanics PHYS 212B

Problem Set 1

Due Tuesday, January 12, 2016

Exercise 1.1 Consider two identical linear oscillators each with spring constant k. Additionally, those oscillators are coupled by an interaction term $a\ x_1 \cdot x_2$, where x_1 and x_2 are the oscillator (displacement) variables, and a is a constant. Find the energy levels for this system. Hint: transform to new coordinates.

Solution 1.1 The Hamiltonian of the system is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}k(x_1^2 + x_2^2) + ax_1x_2$$

Perform coordinate transformation to get rid of the cross term ax_1x_2 with

$$x_1 = \frac{\sqrt{2}}{2}(x_1' + x_2'), \qquad x_2 = \frac{\sqrt{2}}{2}(x_1' - x_2'),$$

and p's also satisfy the same transformation

$$p_1 = \frac{\sqrt{2}}{2}(p_1' + p_2'), \qquad p_2 = \frac{\sqrt{2}}{2}(p_1' - p_2').$$

The Hamiltonian can be expressed as

$$H = \frac{p_1'^2}{2m} + \frac{p_2'^2}{2m} + \frac{1}{2}(k+a)x_1'^2 + \frac{1}{2}(k-a)x_2'^2$$

which is composed by two independent oscillators with different spring constants $k \pm a$, respectively, with angular frequencies

$$\omega_1 = \sqrt{\frac{k+a}{m}} \text{ and } \omega_2 = \sqrt{\frac{k-a}{m}}$$

where $|a| \leq k$. Therefore the energy level will be

$$E_{n_1+n_2} = \left(\frac{1}{2} + n_1\right)\hbar\omega_1 + \left(\frac{1}{2} + n_2\right)\hbar\omega_2$$

where $n_1 = 0, 1, 2, \ldots$, $n_1 = 0, 1, 2, \ldots$

Exercise 1.2 Consider an operator A obeying the commutation relations,

$$[A, J_z] = \frac{1}{2}A$$

$$[[A, \mathbf{J}^2], \mathbf{J}^2] = \frac{1}{2}(A\mathbf{J}^2 + \mathbf{J}^2A) + \frac{3}{16}A$$

where $\mathbf{J}$ is the angular momentum of a system. Operators such as A arise in the decay of systems which emit particles of half-integral spin. By employing these commutation relations find "selection rule" for the operator A in a matrix representation which makes the z-component of angular momentum J_z and $\mathbf{J}^2$ diagonal (corresponding eigenvlues of m and $j(j+1)$, respectively). In other words, which matrix elements $\langle j'm'|A|jm\rangle$ can be non-zero? (To simplify things let $X_j = j(j+1)$.)

Solution 1.2 For J_z and $\mathbf{J}^2$, we have

$$J_z|jm\rangle = m|jm\rangle, \quad \mathbf{J}^2|jm\rangle = j(j+1)|jm\rangle.$$

Using the first commutation relation,

$$\langle j'm'|[A, J_z]|jm\rangle = \frac{1}{2}\langle j'm'|A|jm\rangle$$

with $J_z|jm\rangle = m|jm\rangle$, the equation above can be simplified as

$$(m - m' - \frac{1}{2})\langle j'm'|A|jm\rangle = 0$$

Therefore, the selection rule for m is $\Delta m = 1/2$.

Using the second commutation relation, we have

$$\langle j'm'|[[A, \mathbf{J}^2], \mathbf{J}^2]|jm\rangle = \langle j'm'|\frac{1}{2}(A\mathbf{J}^2 + \mathbf{J}^2 A) + \frac{3}{16}A|jm\rangle$$

with $\mathbf{J}^2|jm\rangle = X_j|jm\rangle$, the equation above can simplified as

$$\left[(X_j - X_{j'})^2 - \frac{1}{2}(X_j + X_{j'}) - \frac{3}{16}\right]\langle j'm'|A|jm\rangle = 0$$

Then, the selection rule is

$$(X_j - X_{j'})^2 - \frac{1}{2}(X_j + X_{j'}) - \frac{3}{16} = 0$$

Exercise 1.3 An electron is initially spin up along the $\hat{z}$-axis. This state is then rotated by an angle θ about an axis $\hat{\theta}$, where $\vec{\theta} = \theta\hat{\theta} = \theta_x\hat{x} + \theta_y\hat{y} + \theta_z\hat{z}$ and $\theta = \sqrt{\vec{\theta}\cdot\vec{\theta}}$.

(a) What is the probability (not amplitude) that the electron is still spin up along the original $\hat{z}$-axis? Spin down along this axis?

(b) Now suppose that an electron initially spin up along the $\hat{z} - axis$ is sequentially rotated first by θ_z about $\hat{z}$, then by θ_y about $\hat{y}$, and finally by θ_x about $\hat{x}$. What is the probability that the electron is spin up along the original $\hat{z}$-axis?

Solution 1.3

(a) Consider rotation operator

$$\begin{aligned} R(\theta) &= \exp\left[-\frac{\vec{\theta}\cdot\vec{\sigma}}{2}\right] \\ &= \cos(\theta/2)I - i\sin(\theta/2)\hat{\theta}\cdot\hat{\sigma} \\ &= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\frac{\theta_z}{\theta} & -i\sin\left(\frac{\theta}{2}\right)\left(\frac{\theta_x}{\theta} - i\frac{\theta_y}{\theta}\right) \\ -i\sin\left(\frac{\theta}{2}\right)\left(\frac{\theta_x}{\theta} + i\frac{\theta_y}{\theta}\right) & \cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\frac{\theta_z}{\theta} \end{pmatrix} \end{aligned}$$

The amplitude that find electron spinning up

$$\langle\uparrow|R(\theta)|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\frac{\theta_z}{\theta} & -i\sin\left(\frac{\theta}{2}\right)\left(\frac{\theta_x}{\theta} - i\frac{\theta_y}{\theta}\right) \\ -i\sin\left(\frac{\theta}{2}\right)\left(\frac{\theta_x}{\theta} + i\frac{\theta_y}{\theta}\right) & \cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\frac{\theta_z}{\theta} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\frac{\theta_z}{\theta},$$

and then

$$P(\uparrow) = |\langle \uparrow |R(\theta)| \uparrow \rangle|^2 = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\frac{\theta_z^2}{\theta^2}$$

Similar for

$$P(\downarrow) = |\langle \downarrow |R(\theta)| \uparrow \rangle|^2 = \sin^2\left(\frac{\theta}{2}\right)\frac{\theta_x^2 + \theta_y^2}{\theta^2}$$

(b) The rotation operator becomes

$$R = e^{i\sigma_x\theta_x}e^{i\sigma_y\theta_y}e^{i\sigma_z\theta_z}$$

Let us calculate $e^{i\sigma_x\theta_x}e^{i\sigma_y\theta_y}$ first,

$$\begin{aligned}
e^{i\sigma_x\theta_x}e^{i\sigma_y\theta_y} &= \left[\cos\left(\frac{\theta_x}{2}\right) + i\sin\left(\frac{\theta_x}{2}\right)\sigma_x\right]\left[\cos\left(\frac{\theta_y}{2}\right) + i\sin\left(\frac{\theta_y}{2}\right)\sigma_y\right] \\
&= \cos\left(\frac{\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right)I + i\sin\left(\frac{\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right)\sigma_x + i\cos\left(\frac{\theta_x}{2}\right)\sin\left(\frac{\theta_y}{2}\right)\sigma_y - i\sin\left(\frac{\theta_x}{2}\right)\sin\left(\frac{\theta_y}{2}\right)\sigma_z \\
&= \begin{pmatrix} \cos\left(\frac{\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right) - i\sin\left(\frac{\theta_x}{2}\right)\sin\left(\frac{\theta_y}{2}\right) & i\sin\left(\frac{\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right) + \cos\left(\frac{\theta_x}{2}\right)\sin\left(\frac{\theta_y}{2}\right) \\ i\sin\left(\frac{\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right) - \cos\left(\frac{\theta_x}{2}\right)\sin\left(\frac{\theta_y}{2}\right) & \cos\left(\frac{\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right) + i\sin\left(\frac{\theta_x}{2}\right)\sin\left(\frac{\theta_y}{2}\right) \end{pmatrix}
\end{aligned}$$

where $\sigma_x\sigma_y = i\sigma_z$ is used. We know $|\uparrow\rangle$ is the eigenstate of z rotation,

$$e^{i\sigma_z\theta_z}|\uparrow\rangle = \left[\cos\left(\frac{\theta_z}{2}\right) - i\sin\left(\frac{\theta_z}{2}\right)\right]|\uparrow\rangle$$

Then

$$\begin{aligned}
\langle \uparrow |e^{i\sigma_x\theta_x}e^{i\sigma_y\theta_y}e^{i\sigma_z\theta_z}|\uparrow\rangle &= \left[\cos\left(\frac{\theta_z}{2}\right) - i\sin\left(\frac{\theta_z}{2}\right)\right]\left[\cos\left(\frac{\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right) - i\sin\left(\frac{\theta_x}{2}\right)\sin\left(\frac{\theta_y}{2}\right)\right] \\
\langle \downarrow |e^{i\sigma_x\theta_x}e^{i\sigma_y\theta_y}e^{i\sigma_z\theta_z}|\uparrow\rangle &= \left[\cos\left(\frac{\theta_z}{2}\right) - i\sin\left(\frac{\theta_z}{2}\right)\right]\left[i\sin\left(\frac{\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right) - \cos\left(\frac{\theta_x}{2}\right)\sin\left(\frac{\theta_y}{2}\right)\right]
\end{aligned}$$

The final probability

$$\begin{aligned}
P(\uparrow) =& |\langle \uparrow |e^{i\sigma_x\theta_x}e^{i\sigma_y\theta_y}e^{i\sigma_z\theta_z}|\uparrow\rangle|^2 = \cos^2\left(\frac{\theta_x}{2}\right)\cos^2\left(\frac{\theta_y}{2}\right) + \sin^2\left(\frac{\theta_x}{2}\right)\sin^2\left(\frac{\theta_y}{2}\right) \\
P(\downarrow) =& |\langle \downarrow |e^{i\sigma_x\theta_x}e^{i\sigma_y\theta_y}e^{i\sigma_z\theta_z}|\uparrow\rangle|^2 = \sin^2\left(\frac{\theta_x}{2}\right)\cos^2\left(\frac{\theta_y}{2}\right) + \cos^2\left(\frac{\theta_x}{2}\right)\sin^2\left(\frac{\theta_y}{2}\right)
\end{aligned}$$