

**PHYSICS 200B : CLASSICAL MECHANICS
FINAL EXAMINATION**

(1) Consider the nonlinear oscillator described by the Hamiltonian

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 + \frac{1}{4}\epsilon a q^4 + \frac{1}{4}\epsilon b p^4 \quad ,$$

where ϵ is small.

- (a) Find the perturbed frequencies $\nu(J)$ to lowest nontrivial order in ϵ .
- (b) Find the perturbed frequencies $\nu(A)$ to lowest nontrivial order in ϵ , where A is the amplitude of the q motion.
- (c) Find the relationships $\phi = \phi(\phi_0, J_0)$ and $J = J(\phi_0, J_0)$ to lowest nontrivial order in ϵ .

(2) Consider the forced modified van der Pol equation,

$$\ddot{x} + \epsilon(x^4 - 1)\dot{x} + x = \epsilon f_0 \cos(t + \epsilon \nu t) \quad ,$$

where ϵ is small. Carry out the multiple scale analysis to order ϵ . Following §3.3.2 in the Lecture Notes, find and analyze the equation which relates the amplitude A , detuning ν , and force amplitude f_0 for entrained oscillations. Perform the requisite linear stability analysis and make a plot similar to that in Fig. 3.4 of the Lecture Notes. Is there a region of entrained oscillations which exhibits hysteresis as the detuning parameter is varied? If so, find the corresponding range of f_0 over which this occurs.

Bonus: Use Mathematica or Matlab to integrate the equation, showing examples of entrained and heterodyne behavior, as in Fig. 3.6 (1000 Quatloos extra credit).

(3) Consider shock formation in the inviscid Burgers' equation, $c_t + c c_x = 0$. Let the function $c(\zeta) = c(x = \zeta, t = 0)$ be given by the triangular profile,

$$c(\zeta) = c_0 \left(\frac{a - |\zeta|}{a} \right) \Theta(a - |\zeta|) \quad .$$

- (a) Find the break time t_B .
- (b) Implement the shock fitting protocol and find $\zeta_-(t)$, $\zeta_+(t)$, and $x_s(t)$.
- (c) Find the shock discontinuity $\Delta c(t)$ for $t > t_B$.
- (d) Sketch $c(x, t)$ vs. x for $t/t_B = 0, \frac{1}{2}, 1, 2$, and 4. Show that for $t \geq t_B$, $c(x, t)$ vs. x has the form of a right triangle whose area is given by $\int_{-a}^a d\zeta c(\zeta)$.
- (e) Without shock fitting, sketch the characteristics in the (x, t) plane and highlight the region where they cross. Then sketch the characteristics after shock fitting. *Hint: Your sketches should roughly resemble those in Fig. 4.13 of the Lecture Notes.*