Physics 4E | Quiz 4 Solutions | Spring 2016

Problem 1

$$E_{n_1 n_2 n_3} = \frac{\hbar^2 \pi^2}{2m_e}\left(\frac{n_1^2+n_2^2}{L^2} + \frac{n_3^2}{L_3^2}\right)$$

$L = 4\,\text{Å}$. For example, assume the triply degenerate states are:

$(1,2,1)$; $(2,1,1)$; $(1,1,3)$. The condition is:

$$\frac{2^2+1^2}{L^2} + \frac{1^2}{L_3^2} = \frac{1^2+1^2}{L^2} + \frac{9}{L_3^2} \Rightarrow \frac{5}{L^2} + \frac{1}{L_3^2} = \frac{2}{L^2} + \frac{9}{L_3^2} \Rightarrow$$

$$\Rightarrow \frac{3}{L^2} = \frac{8}{L_3^2} \Rightarrow \frac{L_3^2}{L^2} = \frac{8}{3} \Rightarrow \boxed{L_3 = \sqrt{\frac{8}{3}}\,L = 6.532\,\text{Å}} \text{ (a)}$$

So $\frac{1}{L_3^2} = \frac{3}{8}\frac{1}{L^2} = \frac{0.375}{L^2}$ $\Rightarrow$ energies are:

$$E_{n_1 n_2 n_3} = \frac{\hbar^2 \pi^2}{2m_e L^2}\left(n_1^2 + n_2^2 + 0.375\, n_3^2\right)$$

the quantum numbers for the lowest 6 energy states are:

n_1	n_2	n_3	$n_1^2+n_2^2+0.375n_3^2$
2	1	2	6.5
1	2	1	5.375
2	1	1	5.375
1	1	3	5.375
1	1	2	3.5
1	1	1	2.375

	degeneracy
(2,1,2)(1,2,2)	2
(2,1,1)(1,2,1)(1,1,3)	3
(1,1,2)	1
(1,1,1)	1

(b)

(c) $$E_{n_1 n_2 n_3} = \frac{\hbar^2 \pi^2}{2m_e L^2}\left(n_1^2 + n_2^2 + \frac{n_3^2}{100^2}\right)$$ since $L_3 = 100L$

Since the last term is so small, lowest 6 states are (111)(112)(113)(114)(115)(116)

All energies are approximately same, since last term is at most $\frac{36}{10{,}000} = 0.0036$ compared with $1^2+1^2 = 2$ for first 2 terms.

$$E = \frac{\hbar^2}{2m_e}\left(\frac{\pi}{L}\right)^2(2) = 3.81\,\text{eV}\left(\frac{\pi}{4}\right)^2 \cdot 2 = \boxed{4.70\,\text{eV}} \text{ (c)}$$

Problem 2

$$\psi(r,\theta,\phi) = C r e^{-2r/a_0} \sin\theta \, g(\phi)$$

From ($\sin\theta$) we infer $\boxed{l=1, m=\pm 1} \Rightarrow \boxed{g(\phi) = e^{i\phi} \text{ or } g(\phi) = e^{-i\phi}}$

Then, $n \geq 2$. From the r-dependence (no nodes) $\Rightarrow \boxed{n=2}$

(b) The exponential is e^{-Zr/na_0}, $n=2 \Rightarrow \boxed{Z=4}$

energy $E = -E_0 \frac{Z^2}{n^2} = -E_0 \frac{4^2}{2^2} = -4E_0 = \boxed{-54.4\,eV}$

(c) Radial probability:

$$P(r) \propto r^2 R(r)^2 = r^2 \cdot r^2 \cdot e^{-4r/a_0} = r^4 e^{-4r/a_0}$$

$$\langle r \rangle = \frac{\int_0^\infty dr\, r P(r)}{\int_0^\infty dr\, P(r)} = \frac{\int_0^\infty dr\, r^5 e^{-4r/a_0}}{\int_0^\infty dr\, r^4 e^{-4r/a_0}} = \frac{5! \cdot a_0^6 \cdot 4^5}{4^6 \;\; 4! \cdot a_0^5} = \frac{5}{4} a_0$$

So $\boxed{\langle r \rangle = \frac{5}{4} a_0 = 1.25\, a_0}$

$$\langle \tfrac{1}{r} \rangle = \frac{\int_0^\infty dr\, \frac{1}{r} P(r)}{\int_0^\infty dr\, P(r)} = \frac{\int_0^\infty dr\, r^3 e^{-4r/a_0}}{\int_0^\infty dr\, r^4 e^{-4r/a_0}} = \frac{3!\, a_0^4 \cdot 4^5}{4^4 \cdot 4! \cdot a_0^5} = \frac{1}{a_0}$$

$\Rightarrow \boxed{\langle \frac{1}{r} \rangle = \frac{1}{a_0}}$

(d) Maximize $P(r)$: $0 = P'(r_m) = 4r_m^3 - \frac{4}{a_0} r_m^4 = 0 \Rightarrow \boxed{r_m = a_0}$

Most probable radius for electron is $\boxed{r_m = a_0}$

Note that $\boxed{\langle \frac{1}{r} \rangle = \frac{1}{r_m}}$ that is so for all states with $l = n-1$

And $r_m = n^2 \frac{a_0}{Z}$ as in the Bohr atom

Problem 3

Orbital magnetic moment: $\vec{\mu}_\ell = -\frac{e}{2m_e}\vec{L}$

Spin magnetic moment: $\vec{\mu}_s = -\frac{e}{2m_e} g_s \vec{S}$, $g_s = 2$

Total magnetic moment: $\vec{\mu} = \vec{\mu}_\ell + \vec{\mu}_s = -\frac{e}{2m_e}(\vec{L} + 2\vec{S})$

z component: $\mu_z = -\frac{e\hbar}{2m_e}(m + 2m_s) = -\mu_B(m + 2m_s)$

Energy in a magnetic field: $\Delta E = -\vec{\mu}\cdot\vec{B} = -\mu_z B = \mu_B(m + 2s)B$

$\mu_B = 5.79\times10^{-5}\,eV/T$, $B = 69.1\,T \Rightarrow \mu_B B = 4.00\times10^{-3}\,eV$

$\underline{\ell=1:}$ $m = 1, 0, -1$, $2s = \pm 1 \Rightarrow m + 2s = 2, 1, 0, -1, -2 \Rightarrow$

$$\Delta E = \begin{cases} 2\mu_B B = 8\times10^{-3}\,eV & m = 1,\ m_s = 1/2 \\ \mu_B B = 4\times10^{-3}\,eV & m = 0,\ m_s = 1/2 \\ 0 = 0 & m = 1,\ m_s = -1/2 \text{ or } m = -1,\ m_s = 1/2 \\ -\mu_B B = -4\times10^{-3}\,eV & m = 0,\ m_s = -1/2 \\ -2\mu_B B = -8\times10^{-3}\,eV & m = -1,\ m_s = -1/2 \end{cases}$$

6 states

$\boxed{\text{total} = 8 \text{ states}}$

$\underline{\ell=0}$: $m = 0$, $2s = \pm 1$

$$\Delta E = \begin{cases} \mu_B B = 4\times10^{-3}\,eV & m = 0,\ m_s = 1/2 \\ -\mu_B B = -4\times10^{-3}\,eV & m = 0,\ m_s = -1/2 \end{cases}$$

2 states

(b) With spin-orbit coupling, quantum numbers are ℓ, j, m_j with $|\ell - s| \le j \le |\ell + s|$; $-j \le m_j \le j$; energy depends on ℓ and j, not on m_j (if no magnetic field applied)

$\ell = 0 \Rightarrow |\ell - s| = |\ell + s| = 1/2 = j$. $\ell = 1 \Rightarrow j = \frac{1}{2}$ or $j = \frac{3}{2}$

Call the energies $E_{\ell, j}$

$E_{0,\frac{1}{2}}$

$\ell = 0, j = 1/2, m_j = \frac{1}{2}$

$\ell = 0, j = 1/2, m_j = -\frac{1}{2}$

$\boxed{2 \text{ states}}$

$E_{1,\frac{1}{2}}$

$\ell = 1, j = \frac{1}{2}, m_j = \frac{1}{2}$

$\ell = 1, j = \frac{1}{2}, m_j = -\frac{1}{2}$

$\boxed{2 \text{ states}}$

$E_{1,3/2}$

$\ell = 1, j = \frac{3}{2}$, $m_j = 3/2,\ 1/2,\ -1/2,\ -3/2$

$\boxed{4 \text{ states}}$

$\boxed{8 \text{ states}}$

3 different energies $E_{\ell, j}$

$\boxed{E_{1,1/2} < E_{0,1/2} < E_{1,\frac{3}{2}}}$