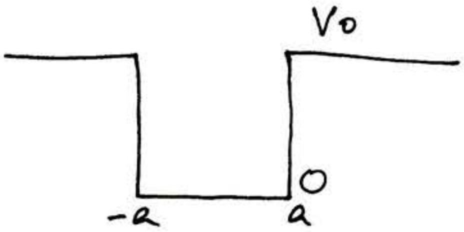
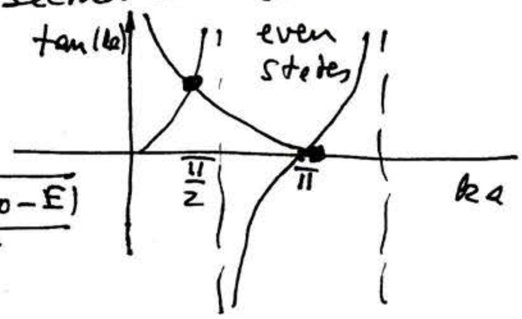
Physics 4E — **Quiz 3 Solutions** — Spring 2016

Problem 1



three bound states ⇒ even, odd, even. The second even state has energy right below V_0.



Condition for even states:

$$\tan(ka) = \frac{\alpha}{k}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}, \quad \alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

Condition is: $V_0 = E_3 \Rightarrow \alpha = 0$

$$ka = \pi = \sqrt{\frac{2mE_3}{\hbar^2}}\cdot a \Rightarrow \frac{2mE_3}{\hbar^2} = \frac{\pi^2}{a^2} \Rightarrow E_3 = \frac{\hbar^2\pi^2}{2ma^2}$$

$$a = 3\text{Å},\ \frac{\hbar^2}{2m} = 3.81\,\text{eV}\,\text{Å}^2 \Rightarrow E_3 = \frac{\hbar^2\pi^2}{2ma^2} = \frac{3.81\pi^2}{9}\,\text{eV} = 4.18\,\text{eV}$$

⇒ **the well should be at least 4.18 eV high**

(b) For the proton, this well is very deep. Assume an ∞ well, find how many states have energy less than $V_0 = 4.18$ eV

$$E_n = \frac{\hbar^2\pi^2}{2m_p(2a)^2}n^2 < V_0 \Rightarrow n^2 < \frac{2m_p(2a)^2 V_0}{\hbar^2\pi^2} = 7350 = 85.7^2$$

Approximately 85 states

(c) If the well was 4.18 eV for the electron it would be like an infinite well (almost), with energy

$$E_3 = \frac{\hbar^2\pi^2}{2mL^2}\times 3^2 = 9.4\,\text{eV}$$

Problem 2

From Appendix B-1: $\int_{-\infty}^{\infty} dx\, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$, $\int_{-\infty}^{\infty} dx\, x^2 e^{-\lambda x^2} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}}$

$\psi(x) = A e^{-\gamma x^2}$

Normalization:

$$1 = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = \int_{-\infty}^{\infty} dx\, A^2 e^{-2\gamma x^2} = A^2 \sqrt{\frac{\pi}{2\gamma}} = 1 \Rightarrow \boxed{A = \left(\frac{2\gamma}{\pi}\right)^{1/4}}$$

(b) $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$; $\langle x\rangle = 0$ by symmetry

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, x^2 |\psi(x)|^2 = \int_{-\infty}^{\infty} dx\, A^2 x^2 e^{-2\gamma x^2} = A^2 \cdot \frac{1}{2} \cdot \frac{1}{2\gamma} \cdot \left(\frac{\pi}{2\gamma}\right)^{1/2} = \frac{1}{4\gamma}$$

note that $A^2 \cdot \left(\frac{\pi}{2\gamma}\right)^{1/2} = 1$

$$\Rightarrow \boxed{\Delta x = \frac{1}{2\sqrt{\gamma}}}$$

(c) $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$; $\langle p\rangle = 0$ by symmetry

$$p_{op}\psi = \frac{\hbar}{i}\frac{d}{dx}\psi = -\frac{\hbar}{i}(2\gamma) x e^{-\gamma x^2} \cdot A$$

$$p_{op}^2 \psi = \frac{\hbar}{i}\frac{d}{dx}\left(\frac{\hbar}{i}\frac{d}{dx}\psi\right) = A\cdot\hbar^2 (2\gamma)\frac{d}{dx}\left(x e^{-\gamma x^2}\right) = A\hbar^2\left(2\gamma - 4\gamma^2 x^2\right) e^{-\gamma x^2}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} dx\, \psi^* p_{op}^2 \psi = A^2\hbar^2 \int_{-\infty}^{\infty} dx\, (2\gamma - 4\gamma^2 x^2) e^{-2\gamma x^2} =$$

$$= A^2\hbar^2\left[2\gamma\sqrt{\frac{\pi}{2\gamma}} - 4\gamma^2 \cdot \frac{1}{2} \cdot \frac{1}{2\gamma}\sqrt{\frac{\pi}{2\gamma}}\right] = \hbar^2[2\gamma - \gamma] = \hbar^2\gamma$$

$$\Rightarrow \boxed{\Delta p = \hbar\sqrt{\gamma}}$$

(d) $\boxed{\Delta x \Delta p = \frac{1}{2\sqrt{\gamma}} \cdot \hbar\sqrt{\gamma} = \frac{\hbar}{2}}$ agrees with uncertainty principle

Problem 3

$\psi(x) = e^{ik_1x} + Be^{-ik_1x} \qquad x<0$

$\psi(x) = 1.5e^{ik_2x} \qquad x>0$

(a) Continuity at $x=0 \Rightarrow 1+B=1.5 \Rightarrow \boxed{B=0.5}$

(b) Reflection coefficient is $R = \left|\frac{B}{A}\right|^2 = 0.5^2 = 0.25$

$\Rightarrow$ transmission coeff is $T = 1-R = 0.75$

transmitted $= T \times$ incident $= 0.75 \times 1000 = \boxed{750}$

(c) Continuity of derivative:

$k_1(1-B) = 1.5\,k_2 \Rightarrow \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$\Rightarrow 0.5k_1 = 1.5k_2 \Rightarrow k_1 = 3k_2 \qquad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

$\Rightarrow \sqrt{\frac{2mE}{\hbar^2}} = 3\sqrt{\frac{2m}{\hbar^2}(E-V_0)} \Rightarrow \frac{2mE}{\hbar^2} = 9\,\frac{2m}{\hbar^2}(E-V_0)$

$\Rightarrow E = 9E - 9V_0 \Rightarrow 8E = 9V_0 \Rightarrow \boxed{E = \frac{9}{8}V_0 = 9\,eV}$

kinetic energy of incident and reflected electrons 9 eV

kinetic energy of transmitted electrons 1 eV