Quiz 2 Solutions

Problem 1

(a) Rutherford's law first fails for large angles, small impact parameters, because α-particles penetrate nucleus. Fewer particles scatter at large angles than predicted by Rutherford's formula.

(b) Relation between impact parameter b and scattering angle θ (Eq. 4-3 text)

$$b = \frac{k q_\alpha Q}{m_\alpha v^2} \cot\frac{\theta}{2} \Rightarrow \frac{k q_\alpha Q}{m_\alpha v^2} = \frac{b}{\cot\frac{\theta}{2}} = b\tan\frac{\theta}{2}$$

The distance of closest approach (for head on collision) is (for $\theta = 120°$ in)

$$r_d = \frac{k q_\alpha Q}{\frac{1}{2} m_\alpha v^2} = 2b\tan\frac{\theta}{2} = 2b\tan\left(\frac{120°}{2}\right) = 2\sqrt{3}\,b = 2\sqrt{3}\cdot 2.5\cdot 10^{-5}\,\text{Å}$$

$$\Rightarrow \boxed{r_d = 8.66\times 10^{-5}\,\text{Å}}$$

The fact that Rutherford's law fails for large angles for these α-particles means the α-particles are entering the nucleus, hence

$$\boxed{r_{nucleus} > r_d = 8.66\times 10^{-5}\,\text{Å}}$$

(c) $q_\alpha = 2e$, $Q = Ze$. $r_d = \dfrac{2ke^2 Z}{\frac{1}{2} m_\alpha v^2}$ and $\dfrac{m_\alpha v^2}{2} = 7.65\,\text{MeV} \Rightarrow$

$$Z = \frac{r_d}{2ke^2}\cdot\frac{m_\alpha v^2}{2} = \frac{8.66\times 10^{-5}\,\text{Å} \times 7.65\,\text{MeV}}{2\times 14.4\,\text{eV Å}} = \boxed{23}$$

(d) For $\frac{1}{2} m_\alpha v^2 = 7.25\,\text{MeV}$, distance of closest approach is

$$r_d' = r_d\cdot\frac{7.65}{7.25} = 9.14\times 10^{-5}\,\text{Å}$$, Rutherford's law holds for all angles $\Rightarrow$

$r_d' > r_{nucleus} \Rightarrow$ $\boxed{8.66\times 10^{-5}\,\text{Å} < r_{nucleus} < 9.14\times 10^{-5}\,\text{Å}}$

Problem 2

$$L = n\hbar = m v_n r_n \quad , \quad r_n = r_0 n^2 = \frac{a_0}{Z} n^2 \Rightarrow \text{using } a_0 = \frac{\hbar^2}{mke^2}$$

$$\Rightarrow v_n = \frac{n\hbar}{m r_n} = \frac{n\hbar}{m a_0 n^2} \cdot Z = \frac{\hbar Z}{\hbar^2} \frac{ke^2}{n}$$

$$\Rightarrow \boxed{v_n = \frac{ke^2}{\hbar} \frac{Z}{n}}$$

If for He^+ this electron has same speed as electron in $n=1$ state of H $\Rightarrow$ $\boxed{n = 2}$

(a) transition from $n=2$ to $n=1$ for $Z=2$: $E_0 = 13.6\,eV$

$$\frac{hc}{\lambda} = E_0 Z^2 \left(1 - \frac{1}{4}\right) = \frac{3}{4} E_0 Z^2 \Rightarrow$$

$$\Rightarrow \lambda = \frac{4}{3} \frac{hc}{E_0 Z^2} = \frac{1}{3} \cdot \frac{12{,}400}{13.6} \mathring{A} = \boxed{303.92 \mathring{A}}$$

(b) longest wavelength photon is for $n=2 \rightarrow n=3$

$$\frac{1}{2^2} - \frac{1}{3^2} = \frac{1}{4} - \frac{1}{9} = \frac{5}{36} \Rightarrow \lambda = \frac{36}{5} \frac{hc}{E_0 Z^2} = \frac{9}{5} \cdot \frac{12400}{13.6} \mathring{A}$$

$$\Rightarrow \boxed{\lambda = 1641.2 \mathring{A}}$$

(c) Classically, $f^{-1} = \frac{\lambda_{cl}}{c} = \frac{2\pi r}{v} = 2\pi \frac{a_0 n^2}{Z} \frac{\hbar \cdot n}{ke^2 Z} = \frac{2\pi a_0}{Z^2} \frac{\hbar}{ke^2} n^3 \Rightarrow$

$$\Rightarrow \lambda_{cl} = \frac{2\pi a_0}{Z^2} n^3 \frac{\hbar c}{ke^2} = \frac{2\pi a_0}{2^2} \cdot 2^3 \cdot 137 = 4\pi a_0 \cdot 137 = 910.7 \mathring{A}$$

so $\boxed{\lambda_{cl} = 910.7 \mathring{A}}$. The average of the wavelengths for the transitions $n=2 \rightarrow n=1$ and $n=2 \rightarrow n=3$ is

$$\frac{303.92 \mathring{A} + 1641.2 \mathring{A}}{2} = \boxed{973 \mathring{A}}$$ not too far from λ_{cl}.

For larger n, these results will converge because of the correspondence principle.

Problem 3

From the uncertainty principle, for a particle confined to a length L

$$\Delta p \sim \frac{\hbar}{L}$$, so the kinetic energy for an electron is

$$\bar{E}_{kin} = \frac{\bar{p}^2}{2m} = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2mL^2} = \frac{1973^2}{2\times 511{,}000\times 10^{-4}}\ eV \cong \boxed{38{,}089\,eV}$$

(b) For the ground state of a hydrogen-like ion,

$$E_{kin} = E_0 Z^2 \Rightarrow Z \cong \left(\frac{E_{kin}}{E_0}\right)^{1/2} \cong \left(\frac{38{,}089}{13.6}\right)^{1/2} \cong 52.9$$

$$\Rightarrow \boxed{Z = 53}$$

(c) $$r_0 = \frac{a_0}{Z} = \frac{0.529}{53}\ \mathring{A} = 0.00098\ \mathring{A} \sim 10^{-2}\ \mathring{A}$$

The radius is approximately equal to $10^{-2}\ \mathring{A}$ because the kinetic energy of the electron can be understood to arise from the uncertainty principle for the electron being confined in a distance of order r, the radius of the orbit.