6-5. (a) $\Psi(x,t) = A\sin(kx - \omega t)$

$$\frac{\partial \Psi}{\partial t} = -\omega A\cos(kx - \omega t)$$

$$i\hbar\frac{\partial \Psi}{\partial t} = -i\hbar\omega A\cos(kx - \omega t)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = -k^2 A\sin(kx - \omega t)$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \frac{-\hbar k^2 A}{2m}\sin(kx - \omega t) \neq i\hbar\frac{\partial \Psi}{\partial t}$$

(b) $\Psi(x,t) = A\cos(kx - \omega t) + iA\sin(kx - \omega t)$

$$i\hbar\frac{\partial \Psi}{\partial t} = i\hbar\omega A\sin(kx - \omega t) - i^2\hbar\omega A\cos(kx - \omega t)$$

$$= \hbar\omega A\cos(kx - \omega t) + i\hbar\omega A\sin(kx - \omega t)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \frac{\hbar^2 k^2 A}{2m}\cos(kx - \omega t) + \frac{\hbar^2 i k^2 A}{2m}\sin(kx - \omega t)$$

$$= \frac{\hbar^2 k^2}{2m}\left[A\cos(kx - \omega t) + iA\sin(kx - \omega t)\right]$$

$$= i\hbar\frac{\partial \Psi}{\partial t} \quad \text{if } \frac{\hbar^2 k^2}{2m} = \hbar\omega \text{ it does. (Equation 6-5 with } V = 0\text{)}$$

6-9. (a) The ground state of an infinite well is $E_1 = h^2/8mL^2 = (hc)^2/8mc^2L^2$

For $m = m_p$, $L = 0.1nm$: $E_1 = \dfrac{(1240MeV\bullet fm)^2}{8(938.3\times10^6 eV)(0.1nm)^2} = 0.021eV$

(b) For $m = m_p$, $L = 1fm$: $E_1 = \dfrac{(1240MeV\bullet fm)^2}{8(938.3\times10^6 eV)(1fm)^2} = 205MeV$

6-10. The ground state wave function is ($n = 1$) $\psi_1(x) = \sqrt{2/L}\sin(\pi x/L)$ (Equation 6-32)

The probability of finding the particle in Δx is approximately:

$$P(x)\Delta x = \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)\Delta x = \frac{2\Delta x}{L}\sin^2\left(\frac{\pi x}{L}\right)$$

(a) For $x = \frac{L}{2}$ and $\Delta x = 0.002L$, $P(x)\Delta x = \frac{2(0.002L)}{L}\sin^2\left(\frac{\pi L}{2L}\right) = 0.004\sin^2\frac{\pi}{2} = 0.004$

(b) For $x = \frac{2L}{3}$ and $\mathrm{P}(\mathrm{x})\Delta x = \frac{2(0.002L)}{L}\sin^2\left(\frac{2\pi L}{3L}\right) = 0.004\sin^2\frac{2\pi}{3} = 0.0030$

(c) For $x = L$ and $\mathrm{P}(\mathrm{x})\Delta x = 0.004\sin^2\pi = 0$

6-11. The second excited state wave function is ($n = 3$) $\psi_3(x) = \sqrt{2/L}\sin(3\pi x/L)$ (Equation 6-32). The probability of finding the particle in Δx is approximately:

$$P(x)\Delta x = \frac{2}{L}\sin^2\left(\frac{3\pi x}{L}\right)\Delta x$$

(a) For

$x = \frac{L}{2}$ and $\Delta x = 0.002L$, $P(x)\Delta x = \frac{2(0.002L)}{L}\sin^2\left(\frac{3\pi L}{2L}\right) = 0.004\sin^2\frac{3\pi}{2} = 0.004$

(b) For $x = \frac{2L}{3}$ and $\mathrm{P}(\mathrm{x})\Delta x = 0.004\sin^2\left(\frac{6\pi L}{3L}\right) = 0.004\sin^2 2\pi = 0$

(c) For $x = L$ and $\mathrm{P}(\mathrm{x})\Delta x = 0.004\sin^2\left(\frac{3\pi L}{L}\right) = 0.004\sin^2 3\pi = 0$

6-16. $E_n = \frac{h^2 n^2}{8mL^2}$ and $\Delta E_n = E_{n+1} - E_n = \frac{h^2}{8mL^2}(n^2 + 2n + 1)$

or, $\Delta E_n = (2n+1)\frac{h^2}{8mL^2} = \frac{hc}{\lambda}$

so, $L = \left(\frac{3\lambda h}{8mc}\right)^{1/2} = \left(\frac{3\lambda hc}{8mc^2}\right)^{1/2} = \left(\frac{3(694.3nm)(1240eV\bullet nm)}{8(0.511\times10^6 eV)}\right)^{1/2} = 0.795nm$

6-21. (a) For an electron: $E_1 = \frac{(1240MeV\bullet fm)^2}{8(0.511MeV)(10fm)^2} = 3.76\times10^3 MeV$

(b) For a proton: $E_1 = \frac{(1240MeV\bullet fm)^2}{8(938.3Mev)(10fm)^2} = 2.05MeV$

(c) $\Delta E_{21} = 3E_1$ (See Problem 6-16)

For the electron: $\Delta E_{21} = 3E_1 = 1.13\times10^4 MeV$

For the proton: $\Delta E_{21} = 3E_1 = 6.15MeV$

6-25. Refer to MORE section "Graphical Solution of the Finite Square Well". If there are only two allowed energies within the well, the highest energy $E_2 = V_0$, the depth of the well.

From Figure 6-14, $ka = \pi/2$, i.e., $ka = \frac{\sqrt{2mE_2}}{\hbar} \times a = \pi/2$

where $a = 1/2(1.0 fm) = 0.5 fm$ and $m = 939.6 MeV/c^2$ for the neutron.

Substituting above, squaring, and re-arranging, we have:

$$E_2 = V_0 = \left(\frac{\pi}{2}\right)^2 \frac{\hbar^2}{2(939.6 MeV/c^2)(0.5 fm)^2}$$

$$V_0 = \frac{(\pi)^2 (\hbar c)^2}{8(939.6 MeV)(0.5 fm \times 10^{-6} nm/fm)^2} = \frac{(\pi)^2 (197.3 eV \bullet nm)^2}{8(939.6 \times 10^6 eV)(0.5 \times 10^{-6} nm)^2}$$

$$V_0 = 2.04 \times 10^8 eV = 204 MeV$$

6-29. $\langle p_x \rangle = \int_{-\infty}^{+\infty} \psi_3^* \ x \ \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_3 \ s \ dx$ (Equation 6-48)

$$= \int_0^L \sqrt{\frac{2}{L}} \sin\frac{3\pi x}{L} \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) \sqrt{\frac{2}{L}} \sin\frac{3\pi x}{L} dx$$

$$= \frac{2}{L}\frac{\hbar}{i}\int_0^L \left(\sin\frac{3\pi x}{L}\right)\left(\cos\frac{3\pi x}{L}\right)\left(\frac{3\pi}{L}\right) dx$$

Let $\frac{3\pi x}{L} = y$ Then $x = 0 \rightarrow y = 0$, $x = L \rightarrow y = 3\pi$, and $\frac{3\pi}{L}dx = dy \rightarrow dx = \frac{L}{3\pi}dy$

Substituting above gives:

$$\langle p_x \rangle = \frac{2}{L}\frac{\hbar}{i}\frac{L}{3\pi}\int_0^{3\pi} \sin y \cos y dy \times \left(\frac{3\pi}{L}\right)$$

$$= \frac{2}{L}\frac{\hbar}{i}\int_0^{3\pi} \sin y \cos y dy$$

$$= \frac{2}{L}\frac{\hbar}{i}\left(\frac{\sin^2 y}{2}\right)\Bigg|_0^{3\pi} = \frac{2}{L}\frac{\hbar}{i} \ 0 - 0 \ = 0$$

Reconiliation: p_x is a vector pointing half the time in the $+x$ direction, half in the $-x$ direction. E_k is a scalar proportional to v^2, hence always positive.

6-30. For $n = 3$, $\psi_3 = \; 2/L \;^{1/2} \sin \; 3\pi x/L$

(a) $\langle x \rangle = \int_0^L x \;\; 2/L \;\; \sin^2 \; 3\pi x/L \;\; dx$

Substituting $u = 3\pi x/L$, then $x = Lu/3\pi$ and $dx = \;\; L/3\pi \;\; du$. The limits become:

$x = 0 \rightarrow u = 0$ and $x = L \rightarrow u = 3\pi$

$$\langle x \rangle = \;\; 2/L \quad L/3\pi \quad 1/3\pi \;\; \int_0^{3\pi} u \sin^2 u du$$

$$= \;\; 2/L \quad L/3\pi \;\;^2 \left[\frac{u^2}{4} - \frac{u \sin 2u}{4} - \frac{\cos 2u}{8} \right]_0^{3\pi}$$

$$= \;\; 2/L \quad 1/3\pi \;\;^2 \;\; 3\pi \;\;^2 /4 = L/2$$

(b) $\langle x^2 \rangle = \int_0^L x^2 \;\; 2/L \;\; \sin^2 \; 3\pi x/L \;\; dx$

Changing the variable exactly as in (a) and noting that:

$$\int_0^{3\pi} u^2 \sin^2 u du = \left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8} \right) \sin 2u - \frac{u \cos 2u}{4} \right]_0^{3\pi}$$

We obtain $\langle x^2 \rangle = \left(\frac{1}{3} - \frac{1}{18\pi^2} \right) L^2 = 0.328L^2$

6-32. $-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V \;\; x \;\; \psi \;\; x \;\; = E\psi \;\; x$ (Equation 6-18)

$$\frac{1}{2m} \left(\frac{\hbar}{i} \frac{d}{dx} \right) \left(\frac{\hbar}{i} \frac{d}{dx} \right) \psi \;\; x \;\; = \left[E - V \;\; x \;\; \right] \psi \;\; x$$

$$\frac{1}{2m} p_{op} p_{op} \psi = \left[E - V \;\; x \;\; \right] \psi$$

Multiplying by $\psi *$ and integrating over the range of x,

$$\int_{-\infty}^{+\infty} \psi^* \frac{p_{op}^2}{2m} \psi dx = \int_{-\infty}^{+\infty} \psi^* \left[E - V \;\; x \;\; \right] \psi dx$$

$$\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \left[E - V \;\; x \;\; \right] \right\rangle \text{ or } \left\langle p^2 \right\rangle = \left\langle 2m \left[E - V \;\; x \;\; \right] \right\rangle$$

For the infinite square well $V(x) = 0$ wherever $\psi(x)$ does not vanish and vice versa.

Thus, $\langle V(x)\rangle = 0$ and $\langle p^2\rangle = \langle 2mE\rangle = \left\langle 2m\dfrac{n^2\pi^2\hbar^2}{2mL^2}\right\rangle = \dfrac{\pi^2\hbar^2}{L^2}$ for $n = 1$

6-33. $\langle x^2\rangle = \dfrac{L^2}{3} - \dfrac{L^2}{2\pi^2}$ (See Problem 6-30.) And $\langle x\rangle = \dfrac{L}{2}$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \left[\frac{L^2}{3} - \frac{L^2}{2\pi^2} - \frac{L^2}{4}\right]^{1/2} = L\left[\frac{1}{12} - \frac{1}{2\pi^2}\right]^{1/2} = 0.181L$$

$\langle p_x^2\rangle = \dfrac{\pi^2\hbar^2}{L^2}$ and $\langle p\rangle = 0$ (See Problem 6-32)

$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \left[\dfrac{\pi^2\hbar^2}{L^2} - 0\right]^{1/2} = \dfrac{\pi\hbar}{L}$. And $\sigma_x\sigma_p = (0.181L)(\pi\hbar/L) = 0.568\hbar$

6-34. $\psi_0(x) = A_0 e^{-m\omega x^2/2\hbar}$ where $A_0 = (m\omega/\hbar\pi)^{1/4}$

$\langle x\rangle = \displaystyle\int_{-\infty}^{+\infty} A_0^2 x e^{-m\omega x^2/\hbar}dx$ Letting $u^2 = m\omega x^2/\hbar$ and $x = (\hbar/m\omega)^{1/2}u$

$2udu = (m\omega/\hbar)(2xdx)$. And thus, $(m\omega/\hbar)^{-1}udu = xdx$; limits are unchanged.

$\langle x\rangle = A_0^2(\hbar/m\omega)\displaystyle\int_{-\infty}^{+\infty} ue^{-u^2}du = 0$ (Note that the symmetry of $V(x)$ would also tell us that $\langle x\rangle = 0$.)

$$\langle x^2\rangle = \int_{-\infty}^{+\infty} A_0^2 x^2 e^{-m\omega x^2/\hbar}dx$$

$$= A_0^2(\hbar/m\omega)^{3/2}\int_{-\infty}^{+\infty} u^2e^{-u^2}du = 2A_0^2(\hbar/m\omega)^{3/2}\int_{-\infty}^{+\infty} u^2e^{-u^2}du$$

$$= 2A_0^2(\hbar/m\omega)^{3/2}\sqrt{\pi}/4 = (m\omega/\hbar\pi)^{1/2}(\hbar/m\omega)^{3/2}\sqrt{\pi}/2 = \hbar/(2m\omega)$$

6-35. $\dfrac{p^2}{2m} + \dfrac{1}{2}m\omega^2x^2 = (n + 1/2)\hbar\omega$. For the ground state ($n = 0$),

$\langle x^2\rangle = \dfrac{2}{m\omega^2}(\hbar\omega/2 - p^2/2m)$ and $\langle x^2\rangle = \left\langle\dfrac{\hbar}{m\omega} - \dfrac{p^2}{m^2\omega^2}\right\rangle = \hbar/2m\omega$ (See Problem 6-34)

$\dfrac{\hbar}{m\omega}\left\langle 1 - \dfrac{p^2}{m\hbar\omega}\right\rangle = \dfrac{\hbar}{2m\omega}$ or $\left\langle 1 - \dfrac{p^2}{m\hbar\omega}\right\rangle = \dfrac{1}{2} \rightarrow \langle p^2\rangle = \dfrac{1}{2}m\hbar\omega$

6-36. (a) $\Psi_0(x,t) = (m\omega/\hbar\pi)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2}$

(b) $p_{xop} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ $\qquad \langle p^2 \rangle = \int_{-\infty}^{+\infty} \Psi_0^*(x,t)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi_0(x,t)\,dx$

$$\frac{\partial \Psi_0}{\partial x} = A_0(m\omega x/\hbar\pi)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2}$$

$$\frac{\partial^2 \Psi_0}{\partial x^2} = A_0\left[(-m\omega x/\hbar)(-m\omega x/\hbar) - m\omega/\hbar\right] e^{-m\omega x^2/2\hbar} e^{-i\omega t/2}$$

$$\langle p^2 \rangle = -\hbar^2 A_0^2 (m\omega/\hbar)\int_{-\infty}^{+\infty}(m\omega x^2/\hbar - 1)\,e^{-m\omega x^2/\hbar}dx$$

$$= -\hbar^2 A_0^2 (m\omega/\hbar)\left[\int_{-\infty}^{+\infty}(m\omega x^2/\hbar)\,e^{-m\omega x^2/\hbar}dx - \int_{-\infty}^{+\infty} e^{-m\omega x^2/2\hbar}dx\right]$$

Letting $u = (m\omega x/\hbar)^{1/2} x$, then

$$\langle p^2 \rangle = -\hbar^2 A_0^2 (m\omega/\hbar)(m\omega\hbar)^{-1/2}\left[\int_{-\infty}^{+\infty} u^2 e^{-u^2}du - \int_{-\infty}^{+\infty} e^{-u^2}du\right]$$

$$= -\hbar^2 A_0^2 (m\omega/\hbar)^{1/2}\, 2\left[\int_0^{\infty} u^2 e^{-u^2}du - \int_0^{\infty} e^{-u^2}du\right]$$

$$= -\hbar^2 (m\omega/\hbar\pi)^{1/2}(m\omega/\hbar)^{1/2}\, 2\left(\frac{\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2}\right)$$

$$= \hbar^2 (m\omega/\hbar)(1/2) = m\hbar\omega/2$$

6-37. $\psi_0(x) = C_0 e^{-m\omega x^2/2\hbar}$ (Equation 6-58)

(a) $\int_{-\infty}^{+\infty} |\psi_0(x)|^2 dx = 1 = \int_{-\infty}^{+\infty} |C_0|^2 e^{-m\omega x^2/\hbar} dx$

$$= |C_0|^2 \times 2I_0 = |C_0|^2 \times 2 \times \frac{1}{2}\sqrt{\frac{\pi}{x}} \text{ with } \lambda = m\omega/\hbar$$

$$= |C_0|^2 \sqrt{\frac{\pi\hbar}{m\omega}}$$

$$C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

(b) $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi_0|^2 dx = \int_{-\infty}^{+\infty} x^2 \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega x^2/\hbar} dx$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \times 2I_2 = \sqrt{\frac{m\omega}{\pi\hbar}} \times 2 \times \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}} \quad \text{with } \lambda = m\omega/\hbar$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\pi\hbar^3}{m^3\omega^3}} = \frac{1}{2}\frac{\hbar}{\mathrm{m}\omega}$$

(c) $\langle V\ x \rangle = \left\langle \frac{1}{2} m\omega^2 x^2 \right\rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{1}{2} m\omega \times \frac{1}{2}\frac{\hbar}{m\omega} = \frac{1}{4}\hbar\omega$

6-43. $\psi_0\ x\ = A_0 e^{-\omega x^2/2\hbar} \qquad \psi_1\ x\ = A_1\sqrt{\frac{m\omega}{\hbar}} x e^{-m\omega x^2/2\hbar}$

From Equation 6-58.

Note that ψ_0 is an even function of x and ψ_1 is an odd function of x.

It follows that $\int_{-\infty}^{+\infty} \psi_0 \psi_1 dx = 0$

6-53. (a) The probability density for the ground state is $P\ x\ = \psi^2\ x\ =\ 2/L\ \sin^2 \pi x/L$.
The probability of finding the particle in the range $0 < x < L/2$ is:

$$P = \int_0^{L/2} P\ x\ dx = \frac{2}{L}\frac{L}{\pi}\int_0^{\pi/2} \sin^2 u du = \frac{2}{\pi}\left(\frac{\pi}{4} - 0\right) = \frac{1}{2} \quad \text{where } u = \pi x/L$$

(b) $P = \int_0^{L/3} P\ x\ dx = \frac{2}{L}\frac{L}{\pi}\int_0^{\pi/3} \sin^2 u du = \frac{2}{\pi}\left(\frac{\pi}{6} - \frac{\sin 2\pi/3}{4}\right) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.195$

(Note: 1/3 is the classical result.)

(c) $P = \int_0^{3L/4} P\ x\ dx = \frac{2}{L}\frac{L}{\pi}\int_0^{3\pi/4} \sin^2 u du = \frac{2}{\pi}\left(\frac{3\pi}{8} - \frac{\sin 3\pi/2}{4}\right) = \frac{3}{4} + \frac{1}{2\pi} = 0.909$

(Note: 3/4 is the classical result.)