M4A33: Taylor Dispersion – Shear augmented diffusion.

Neither simple diffusion, nor advection by a constant velocity is a particularly effective method of mixing. The former is only rapid when there are sharp concentration gradients, while the latter merely carries material downstream without changing its distribution. A **non-constant** advecting velocity is quite different, however. Then parts of the material are carried faster than others, so that the distribution gets stretched and distorted. This results in sharp gradients perpendicular to the flow, which diffusion can then iron out. The combination of a shearing advection velocity with weak diffusion can lead to quite effective mixing. This is important for gas exchange in the lungs and also in the blood circulation.

We consider steady flow in a straight pipe, driven by a constant pressure gradient ("Poiseuille flow.") In terms of cylindrical polar coordinates $(r,\,\theta,\,z)$, the down-pipe velocity is $(0,\,0,\,V(r))$, where

$$V(r) = 2\overline{V}\left(1-\frac{r^2}{a^2}\right) \qquad \text{where} \quad \overline{V} = \frac{1}{\pi a^2}\int_0^{2\pi} d\theta \int_0^a rV\,dr = \frac{2}{a^2}\int_0^a rV\,dr \tag{1}$$

is the average velocity over the pipe cross-section. In general, we shall denote the average of a quantity X over the pipe by $\overline{X}$. Note that $\overline{X}\,\overline{y} \neq \overline{Xy}$.

Suppose now that an axisymmetric distribution of material, $c(r,\,z,\,t)$, is released into this flow at $t=0$. Then its evolution is described by

$$c_t + V(r)c_z = D\nabla^2 c = D(c_{zz} + \tfrac{1}{r}(rc_r)_r)\ . \tag{2}$$

If no material can enter or leave through the boundary, we must have $\partial c/\partial r = 0$ on $r=a$. We separate c into its cross-sectional average and r-dependent parts, writing

$$c(r,\,z,\,t) = \overline{c}(z,\,t) + c'(r,\,z,\,t) \qquad \text{where} \quad \overline{c} = \frac{2}{a^2}\int_0^a rc\,dr$$

and c' has zero average, $\overline{c'}=0$. Then

$$\overline{c}_t + c'_t + V\overline{c}_z + Vc'_z = D(\overline{c}_{zz} + c'_{zz} + \tfrac{1}{r}(rc'_r)_r)\ . \tag{3}$$

Taking the cross-sectional average of (3), gives

$$\overline{c}_t + \overline{V}\overline{c}_z + \overline{Vc'_z} = D\overline{c}_{zz}\ , \tag{4}$$

where we have used $\partial c'/\partial r = 0$ on $r=a$. The transport of the mean concentration $\overline{c}$ thus depends on the average advection of the r-varying part of c, which we calculate below. Subtracting (4) from (3) yields the r-varying component of (3),

$$c'_t + (V(r)-\overline{V})\overline{c}_z + Vc'_z - \overline{Vc'_z} = D\nabla^2 c'\ . \tag{5}$$

So far this is exact. We now approximate by observing that after a time of order a^2/D we expect cross-pipe diffusion to have almost smoothed out variation in the r-direction. Thus for $t \sim O(a^2/D)$, we expect $\overline{c} \gg c'$. Furthermore, we expect gradients in the r-direction to be greater than those in the z-direction, so that the primary balance is

$$(V(r) - \overline{V})\overline{c}_z \simeq \frac{D}{r}(rc'_r)_r \ . \tag{6}$$

Substituting (1) into (6), we have

$$(rc'_r)_r = \frac{\overline{V}\overline{c}_z}{D}\left(r - \frac{2r^3}{a^2}\right) \ . \tag{7}$$

Now $\overline{c}$ does not depend on r, so that we may integrate (7) twice to find

$$c' = \frac{\overline{V}\overline{c}_z(z,\, t)}{D}\left(\tfrac{1}{4}r^2 - \frac{r^4}{8a^2} + A + B\ln r\right) \ .$$

Since c' must be regular at $r = 0$, we have $B = 0$. Furthermore, c' has zero average, so $\int_0^a ru'\, dr = 0$. This fixes $A = -\frac{1}{12}a^2$, so that

$$c'(r,\, z,\, t) = \frac{\overline{V}a^2}{24D}\overline{c}_z(z,\, t)(6R^2 - 3R^4 - 2) \qquad \text{where} \quad R = \frac{r}{a} \ .$$

Equation (4) requires the term $\overline{V(r)c'_z}$, which is

$$\begin{aligned} \overline{V(r)c'_z} &= \frac{a^2\overline{V}^2}{24D}\overline{c}_{zz}(z,\, t)\int_0^1 2(1 - R^2)(6R^2 - 3R^4 - 2)2R\, dR \\ &= \frac{a^2\overline{V}^2}{12D}\overline{c}_{zz}\left(3 - 2 - 1 + \tfrac{3}{4} - 2 + 1\right) \\ &= -\frac{a^2\overline{V}^2}{48D}\overline{c}_{zz} \ . \end{aligned} \tag{8}$$

Substituting this result into (4), we obtain an advection/diffusion equation for the mean concentration $\overline{c}(z,\, t)$,

$$\overline{c}_t + \overline{V}\overline{c}_z = \left(D + \frac{a^2\overline{V}^2}{48D}\right)\overline{c}_{zz} = D_{\text{eff}}\, \overline{c}_{zz} \ , \tag{9}$$

where D_{eff} is an effective downstream diffusion coefficient,

$$D_{\text{eff}} = \left(D + \frac{a^2\overline{V}^2}{48D}\right) = D\left(1 + \tfrac{1}{48}Pe^2\right) \qquad \text{where} \quad Pe = \frac{a\overline{V}}{D} \ . \tag{10}$$

To summarise these results, after a time of order a^2/D, the concentration will be fairly uniform across the pipe, the material will have moved a distance $\overline{V}t$ down the pipe, and will have spread out in the z-direction a distance $O(\sqrt{D_{\text{eff}}t})$.

The effective diffusion coefficient D_{eff} is slightly paradoxical. It has a minimum value as D varies of $D_{min} = a\overline{V}/\sqrt{12}$ when the **Peclet number** $Pe = \sqrt{48}$. It increases as D decreases from the minimising value. This is because when the cross-pipe diffusion is weak, the shear has a long time to stretch the initial distribution before it is smeared out across the pipe. The length of the mixed region is thus greater.