"How many magnetic field lines in the universe?"

Turbulent Transport

I) Case Study: Transport in Stochastic Fields

A) Review – Basics of Hamiltonian Chaos (cf. Ott, and other supplementary material)

If integrable system, can write:

$$H = H_0(\underline{J})$$

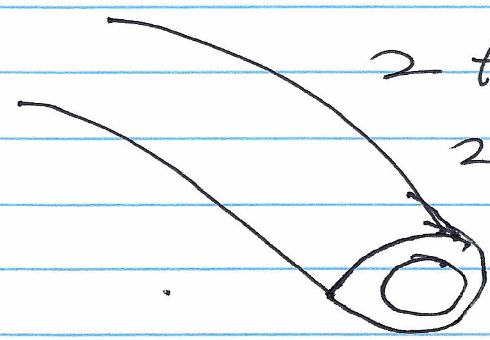
$\underline{J} \equiv$ action variable

$\underline{\theta} \equiv$ angle variable

so

$$\frac{d\underline{\theta}}{dt} = \frac{\partial H}{\partial \underline{J}} = \underline{\omega}(\underline{J})$$

$$\frac{d\underline{J}}{dt} = 0$$



2-torus

2 D-O-F

trajectories lie on toroidal surfaces.

For 2-torus, have:

$\omega_1/\omega_2 = P/Q \rightarrow$ rational number, closed trajectory

$\omega_1/\omega_2 =$ irrational $\rightarrow$ ergodic trajectory, fills surface

recall: Poincaré recurrence....

Surfaces where $\omega_1/\omega_2 = p/q$ are rational surfaces, and define natural resonances of system

Now if perturb:

$$H = H_0(\underline{J}) + \epsilon H_1(\underline{J}, \underline{\theta})$$

then must implement perturbation theory such that canonical structure maintained, so ΔS (correction to action) needed → perturbation in Liouville eqn.

and $\Delta S \sim \epsilon H_1(\underline{J})_{\underline{m}} / \underline{\omega} \cdot \underline{m}$

$\underline{m} \cdot \underline{\omega} = 0$ → "small denominator problem"

→ central issue in chaos theory

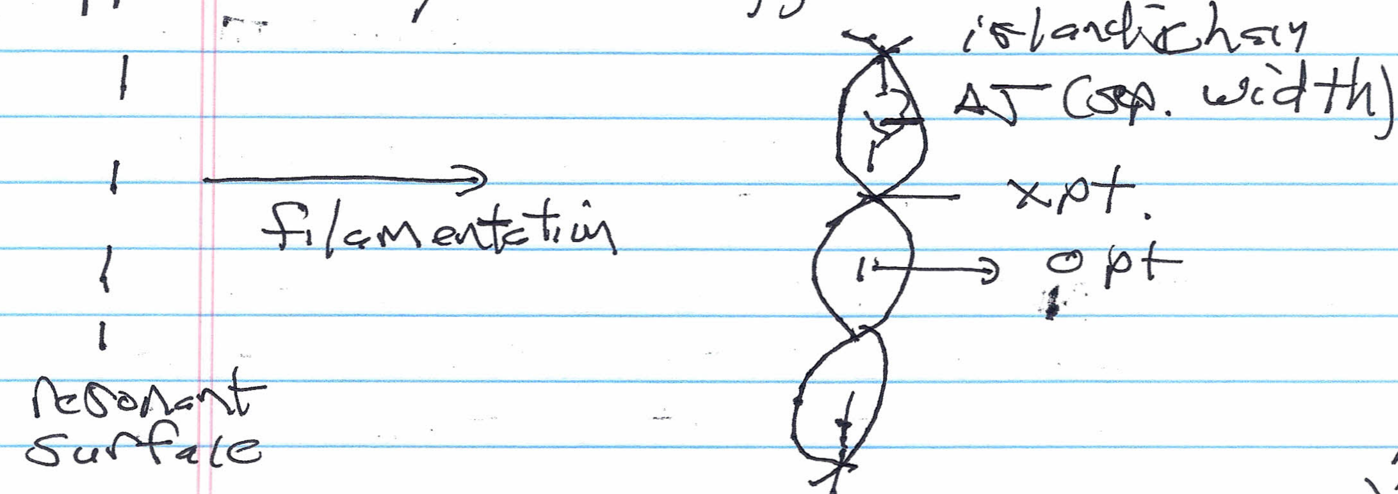
Small denominator problem ⟺ resonance phenomena! (n.b. akin Landau resonance)

i.e. $m\omega_1 + n\omega_2 = 0$

$$m/n = -\omega_2/\omega_1 = -q/p$$

pitch of perturbation — pitch of trajectory

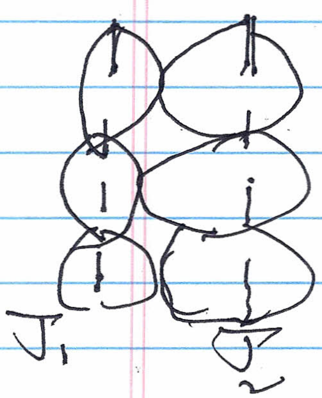
Now, can (for single resonance) resolve small denominator problem by secular perturbation theory (see Supplementary notes), so



lines on perturbed surfaces

$$\Delta J \sim \left(\underbrace{\epsilon H_1}_{\text{perturbation strength}} \Big/ \underbrace{\partial \omega / \partial J}_{\text{shear (diff'nt'l rotation in phase space)}} \right)^{1/2}$$

Now this fix-up works in the region of a single resonance. But if resonances overlap, i.e.



trajectories:
- wander in radius
- fill volume, not surface
- chaos results

Chaos:

- trajectory separation exhibits linear instability, exponentially growing

i.e. $J_1 - J_2 = \Delta J_0 \, e^{\gamma t}$

$\Rightarrow$ 1 (at least) Lyapunov exponent > 0

- chaotic motion $\Rightarrow$ statistical approach for prediction / characterization

$\Rightarrow$ Fokker-Planck Eqn.

or

$\Rightarrow$ Hamiltonian dynamics (Liouville Thm) + chaos

$\therefore$ Quasilinear eqn. ($f \to$ pdf)

(F-P and QLT equiv. for Hamiltonian)

N.B.: Approaches limited to $Ku \ll 1$

- criterion (working) for chaos:

Chirikov overlap:

~~spacing~~ island width

$$\frac{\Delta J_1 + \Delta J_2}{|J_1 - J_2|} > 1$$

spacing

(good working criterion)

i.e. $\dfrac{\Delta W_1 + \Delta W_2}{|\Omega_2 - \Omega_1|} > 1$

islands

- KAM theorem is concerned with ruggedness of irrational surfaces but chaos onset concerned with rational surfaces.

Prime example:

- Magnetic field lines + perturbation

$$\hat{B}_r = \sum_{m,n} B_{m,n,r}\, e^{i(m\theta - n\phi)}$$

- seek D_M → diffusivity of field lines in chaotic regime

but who cares about "lines"? → seek impact on

- heat, particle, momentum transport

and

- is chaotic dynamics always diffusive?

i.e. $Ku \equiv l_{ac}\, \delta B/B \,/\, \Delta r$ <1, >1

$\downarrow$

Kubo #. what of $Ku > 1$?

Line Wandering / Diffusion

if $f = f(r, \theta, z)$ $\rightarrow$ line density
i.e. magnetic flux

then, $\underline{B} \cdot \underline{\nabla} f = 0$

so if $\underline{B} = B_0 \hat{z} + B_\theta(r) \hat{\theta} + \tilde{B}_r \hat{r} + \tilde{B}_\theta \hat{\theta}$

(B_0: toroidal, strong; $B_\theta(r)$: poloidal)

then

$$B_0 \partial_z f + \frac{B_\theta(r)}{r} \partial_\theta f + \underline{\tilde{B}} \cdot \nabla \tilde{f} = 0$$

$$\partial_z f + \frac{B_\theta(r)}{B_0 r} \partial_\theta f + \frac{\underline{\tilde{B}}_\perp}{B_0} \cdot \underline{\nabla} \tilde{f} = 0$$

$$\Rightarrow \partial_z f + \frac{1}{R q(r)} \partial_\theta f + \frac{\underline{\tilde{B}}}{B_0} \cdot \underline{\nabla} f = 0$$

N.B.: $z \to$ plays role of time
- periodicity of fast scale perturbations
- irreversibility of $\langle f \rangle$ evolution

$\theta \to$ periodic

so, for $\langle f \rangle$,

$$\partial_z \langle f \rangle + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle = 0$$

$$\Gamma_{r,B} = \left\langle \frac{\tilde{B}_r \tilde{f}}{B_0} \right\rangle \qquad \text{so} \qquad \text{Fick's Law.}$$

flux of line density

How close?

Now, characteristics of Liouville Eqn.
$\Rightarrow$ equations of lines

$$\frac{dr}{B_r} = \frac{r\,d\theta}{\langle B_\theta(r) \rangle + \tilde{B}_\theta} = \frac{dz}{B_{z0}}$$

so radial excursion given by:

$$dr/dz = \tilde{B}_r/B_0$$

so

$$\delta r \cong \int_0^{\ell} (\tilde{B}_r/B_0)\, dz$$

Now, line trajectory de-coheres from perturbation for $\ell > l_{ac}$
↳ autocorrelation length

$$l_{ac} \cong 1/|\Delta(k_\parallel)|$$ i.e. inverse spatial bandwidth

∴ $$\boxed{\delta r \cong l_{ac}\, \tilde{B}_r/B_0}$$ ⟹ {line excursion in 1 l_{ac}

Can identify $\Delta_r \equiv$ scatterer radial correlation length (i.e. spatial spectral width)

then:

$$K_u \cong \delta r/\Delta_r \cong \frac{l_{ac}}{\Delta_r} \tilde{B}_r/B_0$$ ⟹ Kubo #

and can then posit:

→ $K_u < 1$ ⟹ many kicks in coherence length
⟹ diffusion process

$\left\{ \begin{array}{l} Ku \sim 1 \rightarrow \text{B.B.K. "natural state of EM turbulence"} \\ Ku \sim 1 \rightarrow \text{critical balance.} \end{array} \right.$

$\rightarrow Ku > 1 \rightarrow$ more than one Δ_r in $k_\perp \tilde{b}_r$
$\rightarrow$ strong scattering $\leftrightarrow$ percolation.

Here $Ku < 1$, at first. So, proceed via Quasilinear theory.

$$\Gamma_M = \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle$$

$$= \sum_{\underline{k}} \frac{\tilde{B}_{r-\underline{k}}}{B_0} \tilde{f}_{\underline{k}}$$

$$-i\left(k_z - \frac{k_\theta B_\theta}{B_0}\right)\tilde{f}_{\underline{k}} = -\tilde{B}_{r\underline{k}} \frac{\partial \langle f \rangle}{\partial r}$$

So

$$\Gamma_M = -D_M\, \partial\langle f\rangle/\partial r$$

$$D_M = \sum_{\underline{k}} \left|\frac{\tilde{B}_{r\underline{k}}}{B_0}\right|^2 \pi\delta(k_z - k_\theta B_\theta/B_0)$$

magnetic diffusivity

$$= \sum_{\underline{k}} \left|\frac{\tilde{B}_{r\underline{k}}}{B_0}\right|^2 \pi\,\delta(k_\parallel)$$

(RST 2 '66)

$$\sim \left\langle \left(\frac{\delta B_r}{B_0}\right)^2 \right\rangle l_{ac}$$

N.B.: $\sum_{\underline{n}} = \sum_{m,n}$

$n = \frac{m}{q}$, $dn = \left|\frac{m}{q^2} q'\right| dx$

⇒ spatial scale of spectral width (Δr) sets $|k_\parallel| \sim \left|\frac{k_\theta \Delta r}{L_s}\right|$

$$\boxed{l_{ac} \sim L_s / |k_\theta \Delta r|}$$

Lines then diffuse as:

$$\langle \delta r^2 \rangle \sim D_M z$$

N.B. Line Liouville eqn can be obtained by reducing/simplifying DKE (*)

$$\cancel{\frac{\partial f}{\partial t}} + v_\parallel \hat{n}_0 \cdot \underline{\nabla} f + \cancel{\underline{v}_D \cdot \underline{\nabla} f} - \cancel{\frac{c}{B_0} \underline{\nabla} \phi \times \hat{z} \cdot \nabla f} + v_\parallel \frac{\delta \underline{B}_\perp}{B_0} \cdot \underline{\nabla} f - \cancel{\frac{|e|}{m_e} E_\parallel \frac{\partial f}{\partial v_\parallel}} = \cancel{C(f)}$$

$$\Rightarrow \quad \hat{n}_0 \cdot \nabla f + \frac{\delta B_\perp}{B_0} \cdot \nabla f = 0$$ ✓

Now, scales:

$l_{ac} \Rightarrow$ (scatters)
$\Rightarrow$ field line memory length.

$l_c \Rightarrow$ line decorrelation length

i.e. $$r \frac{d\theta}{dz} = \frac{B_\theta(r)}{B_0}$$

but r scattered, $\Rightarrow$

$$\frac{dy}{dz} = B_\theta(r_0) + \frac{B_\theta'(r_0)}{B_0} \delta r$$

so

$$\frac{d}{dz} \delta y \cong \frac{B_\theta'(r_0)}{B_0} \delta r$$

$$\langle \delta y^2 \rangle = \left\langle \left(\int^z \left(\frac{B_\theta'}{B_0} \right) \delta r \, dz' \right)^2 \right\rangle$$

$\Rightarrow$

$$\langle \delta y^2 \rangle \sim \frac{B_\theta'^2}{B_0^2} z^2 \langle (\delta r)^2 \rangle$$

$$\sim \frac{B_\theta'^2}{B_0^2} D_M z^3$$

again

$$\langle \delta x^2 \rangle \sim D_v \tau^3 \qquad \text{in 1D}$$

For orbit decorrelation length:

$$k_\theta^2 \langle \delta y^2 \rangle \sim k_\theta^2 \frac{B_\theta'^2}{B_0^2} D_M z^3$$

$\Rightarrow$

$$l_c \sim \left(k_\theta^2 \frac{B_\theta'^2}{B_0^2} D_M \right)^{-1/3}$$

$$\sim \left(\frac{k_\theta^2}{L_s^2} D_M \right)^{-1/3}$$

Also

orbit exponentiation length (separation) $\rightarrow$ stretching

show via 2pt. $\langle \delta f(1)\, \delta f(2) \rangle$

→ For QL regime validity:

$l_{ac} < l_c$ → $k u < 1$ (show!)
($l_c < l_{ac}$ → ? prop.)

and another (particle) length: l_{mfp}

$$\boxed{\begin{array}{l} \Rightarrow l_{ac} < l_c < l_{mfp} \rightarrow \text{so called "collisionless regime"} \\ l_{ac} < l_{mfp} < l_c \rightarrow \text{collisional} \end{array}}$$

which brings us to:

⇒ (Electron) <u>Heat</u> Transport

Theme: – waves
– interacting processes

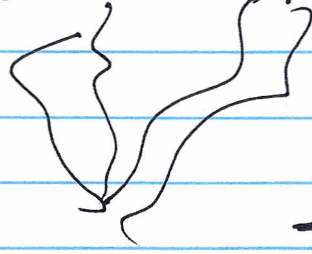
N.B. – nobody cares about "line" diffusion

– people (i.e. experimentalists) do care about:
→ heat
→ particle
→ momentum
} transport.

∴ lets begin with heat transport!

→ Consider $l_{ac} < l_c < l_{mfp}$:

— lines wander



What is $\chi_\perp$?

⇒ "of course its $\chi_\perp \sim v_{Th} D_M$"

→ but is it so simple.

— but, lets assume parallel collisions (only) happen. (Particle stays on line!).

so motion along line is diffusive

$$\delta z^2 \sim D_\parallel t \sim \chi_\parallel t$$

$D_\parallel \to v_{Th}^2/\nu$

$\chi_\parallel \to$ parallel thermal diffusion

→ so: for slug heat:

$$\langle \delta r^2 \rangle \sim D_M z \sim D_M (\chi_\parallel t)^{1/2}$$

$\langle \delta r^2 \rangle \to$ radial scatter

so: $$\chi_\perp = d\langle \delta r^2 \rangle / dt \sim D_M (\chi_\parallel)^{1/2} / t^{1/2}$$

$\to 0$.

Point: → line may wander

but

→ particle kicked back on line

→ even though $l_0 < l_{max}$,

no net radial wander as particle kicked back.

Lessons:

→ collisions control irreversibility

* → need get kicked off field line

→ Need:

— $\perp$ coarse graining:

— FLR → ρ_e

— $\chi_\perp$

— drifts.

minimum resolution scale

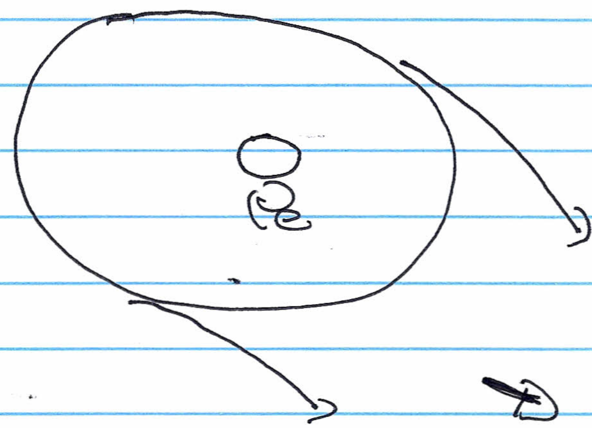
⇒ applied every l_{max}

⇒ coarse graining resets "active volume".

so

→ consider the following argument:

① Consider disk of $r \sim \rho_e$



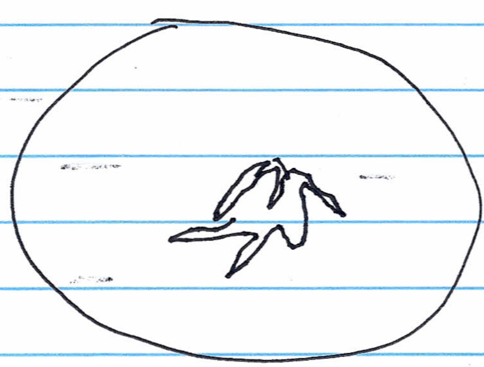
② ⇒

Map disk forward, noting that $\nabla \cdot B = 0$
⇒ map is area preserving

after ~ l_{mfp}

1 $h_L > 0$

1 $h_L < 0$

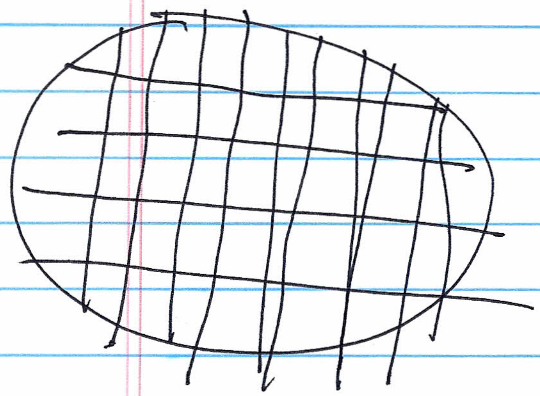


width

$w \sim \rho_e e^{-l_{mfp}/l_c}$

$(l \sim \rho_e e^{+l_{mfp}/l_c})$

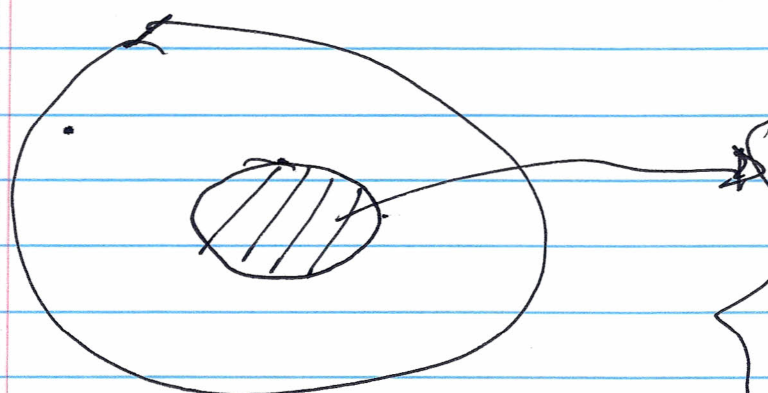
(3) but coarse graining occurs at lmfp



"particle/contour re-sets/smeared" to nearest grid site

so

(4)



coarse graining of structure from previous V.G.

and on continue...

(5) Ludwig Boltzmann assumes no memory between steps (1 lmfp / collision time)

so initial spot expands, with random walk, as

$$\langle \delta r^2 \rangle \sim D_m \, l_{mfp}.$$

i.e. $\boxed{\text{coarse graining interval sets } \langle \delta r^2 \rangle \text{ step!}}$

$\Rightarrow$

⑥ then, for $\chi_\perp$:

$$\chi_\perp \sim \langle \delta r^2 \rangle / \tau_c \sim D_M \frac{l_{mfp}}{\tau_c}$$

$$\sim v_{th} D_M.$$

$$\Rightarrow \boxed{\chi_\perp \sim v_{th} D_M}$$

→ collisionless stochastic field heat diffusivity

→ manifestly independent of collisionality

→ yet clearly dependent on collisions and coarse graining!

Lesson: $\boxed{\text{Coarse graining essential to irreversibility}}$

on

Coarse graining essential to kick particle off field line, or else collisions / back-scatter reverses wander.