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DESTRUCTION OF MAGNETIC SURFACES BY MAGNETIC FIELD IRREGULARITIES

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A perturbation theory analogous to that of quasi-linear theory is developed to treat the behavior of magnetic surfaces, in particular to estimate their stability against field irregularities. The method exploits the similarity between the Vlasov equation and the Liouville equation for field lines. We find that a very important role is played by field resonances, that isolated resonances have a limited effect, extended over a finite width which we estimate, but as soon as resonances overlap, a very rapid destruction of flux surfaces may be expected.

The existence of "magnetic surfaces" in toroidal traps is a crucial ansatz for high temperature plasma confinement in closed system. Such surfaces are known to exist rigorously only in cases of special symmetry, for example, a linear system with helical windings or a torus with a hard core. It is very difficult to estimate the degree to which field lines depart from surfaces in other cases. The mathematical nature of the difficulty lies in the complicated non-linear differential equations which govern the behavior of magnetic field lines and which have coefficients periodically varying along the torus. The problem is to determine whether the solutions of such nonlinear differential equations with periodic coefficients are stable. Especially important perturbations are those which may be in resonance with the rotational transform. In fact, any irregularities of magnetic coils, being periodic along the toroid, might be in resonance, and one may expect a strong influence of such resonant disturbances on the magnetic field lines. On the other hand, strong shear may help to reduce these disturbances because the change of rotational transform with radius restricts the resonances to a small region [*1*].

The purpose of this note is to develop a simple method by which one may approximately treat cases with one, two, or many, resonant perturbations. We find a criterion for "overlapping" of resonances and show that if resonances do not overlap, then flux surfaces are only destroyed in a local region; when resonances overlap strongly, a Brownian motion of flux lines occurs.

To solve the problem we find it convenient to introduce the Liouville equation which has, as its characteristics, the equations for the magnetic field lines. Thus, in cylindrical coordinates we may write, using

$$\mathrm{d}r/\mathrm{d}\varphi = rB_r/B_\varphi$$

$$\mathrm{d}z/\mathrm{d}\varphi = rB_z/B_\varphi$$

$$\frac{\partial f}{\partial \varphi} + \frac{r\,B_r}{B_\varphi}\frac{\partial f}{\partial r} + \frac{r\,B_z}{B_\varphi}\frac{\partial f}{\partial z} = 0 \qquad (1)$$

where now $f=f(r, z, \varphi)$ is constant along a field line, and φ, the coordinate around the torus, plays the role of time in the usual dynamical equations. Note that $f(2\pi)\neq f(0)$ as the field line does not necessarily return to its original position after having gone round the torus. The problem we are now concerned with primarily is a perturbation around a situation in which *exact* flux surfaces exist. We introduce action-angle variables (J, θ) for the unperturbed field; J labels the flux surface and the change in θ following a field line of the unperturbed field is given by $\mathrm{d}\,\theta/\mathrm{d}\,\varphi = \omega(J)$, where ω is related to the rotational transform such that for $\Delta\varphi = 2\pi/\omega$ the field line has returned to its original position on the flux surface. (For example, in a hard-core torus, $J=r^2$, $\theta=\theta^*$; $\omega(J)=(R/r)(B_\theta/B_\varphi)$ but in a helical field the relationship between the angle variable θ and θ^*, the inclination of the line of force, will be a periodic function of φ, θ^*.)

The Liouville equation for field lines now takes the form

$$\frac{\partial f}{\partial t} + \omega(J)\frac{\partial f}{\partial \theta} = \varepsilon\Big[F_J\frac{\partial f}{\partial J} + F_\theta\frac{\partial f}{\partial \theta}\Big] \qquad (2)$$

where $F_J \equiv \mathrm{d}J/\mathrm{d}t$ and $F_\theta \equiv \mathrm{d}\theta/\mathrm{d}t$ as produced by the perturbing field.

As we are primarily interested in the question of behavior of a flux surface originally highly localized in J, we may suppose $\mathrm{d}f/\mathrm{d}J \gg \partial f/\partial\theta$. On the other hand F_J and F_θ, due to the perturbing fields, should be comparable. Hence, in what follows, for simplicity we neglect F_θ and also assume that F_J and $\partial/\partial J$ may be commuted. As the motion will be localized near some resonant J value it will be adequate to describe F_J by

$$F_J = \sum_{n,m} a_{n,m}\,\mathrm{e}^{i(n\Omega t + m\theta)} \qquad (3)$$

where $1/\Omega$ is the periodicity of the torus and $a_{n,m}$ are weak functions of J only. Expanding

$$f = \sum f_l(t)\,\mathrm{e}^{il\theta} \qquad (4)$$

and Laplace-transforming we have

$$-f_l(0) + p\,f_{p,l} + i\,l\,\omega_J f_{p,l} = \varepsilon\sum_{n,m} a_{n,m}\frac{\partial f_{p-in\Omega,\,l-m}}{\partial J} \qquad (5)$$

Here $f_l(0)$, the initial condition, will be taken as $\delta_{l,0}\delta(J-J_0)$ in order to study the "diffusion" of a flux surface, and we may also exclude $n=m=0$ from the sums, these merely relabelling the equilibrium. Iterating eq. 5, we find for $f_{p,0}$ the equation

$$-f(0)+pf_p=\varepsilon^2\sum_{n,m}\sum_{n',m'}a_{n,m}\frac{\partial}{\partial J}\frac{1}{p-in\Omega-im\omega_J}\times a_{n',m'}\frac{\partial f_{p-i(n+n')\Omega,-m-m'}}{\partial J} \quad (6)$$

In order to simplify this further it is necessary to make a "random phase" approximation. In the case of many resonances this can be justified in the usual way. If there are only a few resonances, another justification can be given. The contribution from higher order terms, that is, $n+n'$, $m+m'\neq 0$, may then be estimated by iterating the equation again, which will give a fourth-order term of order

$$\varepsilon^2\frac{\partial}{\partial J}\frac{1}{p+i(\omega-\Omega)}\frac{\partial}{\partial J}\frac{1}{p+i(\omega-\Omega)} \quad (7)$$

times the second order term. The denominators are no smaller than $(\mathrm{d}\omega/\mathrm{d}J)\,\delta J$, even for a resonant case so that these fourth-order terms will be small unless we deal with a case of very small spreading $\mathrm{d}J\approx\varepsilon^{\frac{1}{2}}$. In such cases eq. 5 can only be considered as a qualitative approximation.

Making this approximation we have

$$-\frac{f(0)}{p}+f_p=\varepsilon^2\sum_{n,m}\frac{\partial}{\partial J}\frac{|a_{n,m}|^2}{p^2+(n\Omega-m\omega_J)^2}\frac{\partial f_p}{\partial J} \quad (8)$$

which is evidently very similar to those obtained in the quasi-linear theory where many of these considerations have been given, for example, by Karpman and Altshul [2]. We note, parenthetically, that in higher approximation effects of more complicated resonances can occur analogous to wave-wave coupling but these do not seem of particular importance to this problem.

First we solve for the case of a single resonant perturbation $(n_0,\ m_0)$, where we expand

$$n\,\Omega-m\,\omega_J=m_0\,\omega'\,J$$
$$f(0)=\delta(J-J_0) \quad (9)$$

and put

$$\varepsilon^2|a_{n0,m0}|^2=\eta^2=\text{constant} \quad (10)$$

Then we have

$$-\frac{f(0)]}{p}+f_p=\eta^2\frac{\partial}{\partial J}\frac{1}{p^2+m_0{}^2\omega'^2J^2}\frac{\partial f_p}{\partial J} \quad (11)$$

Putting

$$\frac{1}{p^2+m_0{}^2\omega'^2J^2}\frac{\partial f}{\partial J}\equiv g \quad (12)$$

this can be written

$$-\frac{f'(0)}{p}+(p^2+m_0{}^2\,\omega'^2\,J^2)g=\eta^2\frac{\partial^2 g}{\partial J^2} \quad (13)$$

and is quickly reduced to the Hermite equation, with

$$J=(\eta^2\,u/m_0{}^2\,\omega'^2)^{\frac{1}{4}} \quad (14)$$

namely

$$\frac{\partial^2 g}{\partial u^2}-\left(\frac{p^2}{\eta\,m_0\,\omega'}+u^2\right)g=-\frac{\delta'(u-u_0)}{p\,\eta^2} \quad (15)$$

This may now be solved in the form

$$g=\sum a_n H_n(u) \quad (16)$$

where H_n are the normalized eigenfunctions of the Hermite equation with eigenvalues $2n+1$. We find

$$a_n=\frac{-H_n'(u_0)}{p\,\eta^2[(p^2/\eta\,m_0\,\omega')+2n+1]} \quad (17)$$

and for $f(u,p)$

$$f=\frac{\delta(u-u_0)}{p\left(\frac{\eta}{m_0\,\omega'}\right)^{\frac{1}{2}}}-\sum_n\frac{H_n'(u_0)\,H_n'(u)}{p\left(\frac{\eta}{m_0\,\omega'}\right)^{\frac{1}{2}}\left[\frac{p^2}{\eta\,m_0\,\omega'}+(2n+1)\right]} \quad (18)$$

The Laplace transform can be inverted to give

$$(\eta/m_0\,\omega')^{\frac{1}{2}}f(u)=\delta(u-u_0)-\sum\frac{H_n'(u_0)\,H_n'(u)}{2(2n+1)}[1-\cos(2n+1)^{\frac{1}{2}}(\eta\,m_0\,\omega')\,z]^{\frac{1}{2}} \quad (19)$$

We see that the relevant scale in z is of order $(\eta\,\omega'\,m_0)^{-\frac{1}{2}}$.

We may also consider the steady state solution by returning to eq. 8 and solving for $p=0$, which may be done in terms of Bessel functions, to find, for $u_2>0$

$$f_+=A\,u^{\frac{1}{2}}\,K_{\frac{1}{4}}(\tfrac{1}{2}u^2)\quad(u>u_0)$$
$$f_-=B\,u^{\frac{1}{2}}\,[K_{\frac{1}{4}}(\tfrac{1}{2}u^2)+2^{\frac{1}{2}}\,I_{\frac{1}{4}}(\tfrac{1}{2}u^2)]\quad(u_0>u>0)$$
$$f=B(-u)^{\frac{1}{2}}\,[K_{\frac{1}{4}}(\tfrac{1}{2}u^2)]\quad(u<0) \quad (20)$$

with A and B determined by

$$u_0{}^{-2}[f_+{}'(u_0)-f_-{}'(u_0)]=(\eta/m_0\,\omega')^{-\frac{1}{2}}$$
$$f_+(u_0)=f_-(u_0) \quad (21)$$

The essential point is that the spread in f_0 is determined by

$$(m_0\,\omega'/\eta)\,\delta J^2\sim 1 \quad (22)$$

If

$$(m_0\omega'/\eta)^{\frac{1}{2}}\,J_0>1$$

then

$$\delta J=\eta/m_0\,\delta\omega \quad (23)$$

If

$$(m_0\,\omega'/\eta)^{\frac{1}{2}}\,J_0<1$$

then

$$\delta J\approx(\eta/m_0\,\omega')^{\frac{1}{2}} \quad (24)$$

In order to clarify the meaning of the approximate solution eq. 19, we note that eq. 2 may be solved exactly for the case of a single helical perturbation, yielding a result qualitatively in agreement with that obtained above. In order to see this, let us write the equations of motion for the magnetic field lines in the case of a single resonance as

$$\mathrm{d}r/\mathrm{d}z=\varepsilon\,r^{l-1}\cos(l\theta-k\,z)$$
$$r(\mathrm{d}\theta/\mathrm{d}z)=\Omega(r)-\varepsilon\,r^{l-1}\sin(l\,\theta-k\,z) \quad (25)$$

Equations 25 have the obvious first integral

$$\varepsilon r^l \sin(l\,\theta - k\,z) = \int \{l\,\Omega(r) - k\,r\}\,\mathrm{d}r + \text{constant} \qquad (26)$$

which in the vicinity of the resonant point $r = r_0$, where $l\Omega(r_0) = kr_0$, has the form

$$-\varepsilon_0 r_0{}^l \sin(l\theta - kz) + \{l\,\Omega_0{}'(r_0) - k\}\,(r - r_0)^2 = \text{constant} \qquad (27)$$

Thus, in the case of a single resonance the magnetic surface geometry has the form shown in fig. 1.

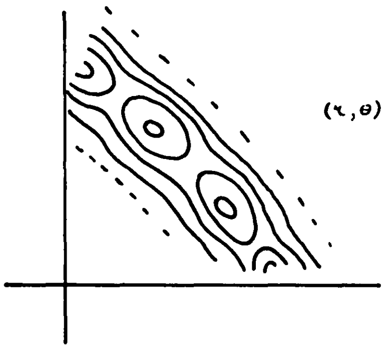

Fig. 1 Magnetic surface geometry for a single resonance.

The resonant perturbation creates subsurfaces spreading over a width $\Delta r \approx (\varepsilon / l\,\omega')^{\frac{1}{2}}$ in the r direction, as found earlier.

An exact solution of eq. 2 for the single resonant case should be a function of the first integrals of eq. 25. This function, corresponding to the initial value $\delta(r - r_0)$, describes the deflections of the magnetic field lines passing through $r = r_0$. From eq. 26 it is obvious that this function is zero for

$$|r - r_0| > (\varepsilon\, r_0{}^l / |l\,\Omega_0{}' - k|)^{\frac{1}{2}} \qquad (28)$$

which shows that the exponentially small tail of the approximate solution eq. 20 should not be believed, due to the neglect of higher order terms in eq. 6.

The advantage of the approximate procedure we have adopted is that it is now easy to see what will happen in the case of several resonances. For example, let us consider the behavior of eq. 8 in the case of two resonances $J = J_1$, J_2 in the long time limit, that is, $p = 0$. Suppose further that the resonances are well spread in the sense that

$$J_1 - J_2 > (\eta / m_0\,\omega')^{\frac{1}{2}} \qquad (29)$$

Then we may solve eq. 8 by the WKB method except very close to the resonances and we see immediately that the solution for f differs from the sum of the individual solutions only by an exponentially small term of order

$$\exp - \frac{(J_1 - J_2)^2\, m_0\, \omega'}{2\eta} \qquad (30)$$

Thus field lines in the neighbourhood of one resonance remain near J_1, and have no opportunity to diffuse away to J_2.

This leads us to our fundamental point. If resonances are spread further apart than $(\eta / m_0\,\omega')^{\frac{1}{2}}$ then no long-term diffusion of field lines can take place but only a small, limited spreading round each resonance. On the other hand, if resonances do overlap then field lines may pass from resonance to resonance.†

For completeness let us evaluate eq. 8 in the limit of many overlapping resonances. Inverting the Laplace transform

$$\frac{\partial f}{\partial t} = \varepsilon^2 \sum_{n,m} \frac{\partial}{\partial J} |a_{n,m}|^2 \int_0^t \cos(n\Omega - m\,\omega_J)(t - t')\, \frac{\partial f(t')}{\partial J}\,\mathrm{d}t' \qquad (31)$$

As we expect the time variation of f to be slow comcompared to ω (at least for small ε) we may treat $\partial f / \partial J$ as constant and find

$$\frac{\partial f}{\partial t} = \varepsilon^2 \sum_m \int \mathrm{d}n \frac{\partial}{\partial J} |a_{n,m}|^2 \frac{\sin(n\Omega - m\,\omega_J)\,t}{n\Omega - m\,\omega_J} \frac{\partial f}{\partial J}$$

$$= \pi\,\varepsilon^2 \sum_m \int \mathrm{d}n \frac{\partial}{\partial J} |a_{n,m}|^2\, \delta(n\,\Omega - m\,\omega_J) \frac{\partial f}{\partial J} \qquad (32)$$

We have replaced the sum over n by an integral in accordance with our assumption of overlapping resonances and replaced $(\sin \alpha x)/x$ by its asymptotic form $\delta(x)$ for α large. Doing the integral, we finally obtain a Fokker-Planck type equation

$$\frac{\partial f}{\partial t} = \pi\,\varepsilon^2 \sum_m \frac{\partial}{\partial J} \left\{ \frac{|a_{n,m}|^2}{\Omega} \right\} \frac{\partial f}{\partial J} \qquad (33)$$

where n is determined by

$$n\,\Omega - m\,\omega_J = 0 \qquad (34)$$

Let us now apply these considerations to a torus with very large rotational transform and large aspect

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† *Note added in proof:*

It might be expected, since any $\omega(J)/\Omega$ is arbitrarily close to some rational value n/m, that all flux surfaces would be affected by resonances and hence diffuse. This is in fact not the case. Let us calculate the probability that a flux surface is not affected by any resonance. We have seen from eq. 24 that the flux surface will be disrupted by resonance (n, m) if

$$\left| \omega_J - \frac{n}{m}\,\Omega \right| < \left(\frac{\eta_{n,m}}{m}\,\omega' \right)^{\frac{1}{2}}$$

Here $\eta_{n,m} \approx$ the n, m Fourier component of the perturbation. Thus the probability for a fixed m that the flux surface will be unaffected is given by

$$P_m = 1 - \frac{1}{\Omega} \sqrt{(m\,\eta_{n,m}\,\omega')}$$

and the total probability that the flux surface remain intact is given by

$$\log P = \sum \ln P_m \approx - \int_1^\infty dm \frac{1}{\Omega} \sqrt{(m\,\eta_{n,m}\,\omega')}$$

Thus if η goes to zero faster than m^{-3} the integral will converge and the high resonances are irrelevant. In fact, however, since the field is governed by Laplace's equation and the sources of field error arise external to the torus, i.e. for $r > r_0$, it follows that

$$\eta_{n,m} \approx \exp(-m)\, \frac{r_0 - r}{r_0}$$

Thus the higher order resonances are not essential.

ratio where the "unperturbed" field may be considered as due to helical windings

$$\cos(l\,\theta^* - k\,\varphi)\ (k \gg 1) \qquad (35)$$

Then two types of perturbation are of importance.

First we have systematic toroidal correction terms of order

$$(r/R)^N \cos N\theta^* \cdot F(l\,\theta^* - k\,\varphi, r) \qquad (36)$$

introduced by the toroidal curvature R. In order to use the present theory, the perturbation 36 must first be expressed in terms of the action-angle variables (J, θ) instead of (r, θ^*). The transformations will be, for example,

$$\theta^* = \theta + G(l\,\theta - k\,\varphi, J) \qquad (37)$$

where G is a periodic function of $(l\theta - k\varphi)$.

It is not necessary to know the actual function G in order to find the resonance points; we need only note that θ^* will contain components

$$\theta + s(l\,\theta - k\,\varphi) \qquad (38)$$

for all integers s. Then the perturbation will contain all components $N\theta + m(l\,\theta - k\varphi)$ so that the resonance condition becomes

$$(N + m\,l)\,\omega(J) = m\,k \qquad (39)$$

It is convenient to represent this in the form

$$l\,\omega(J)/k = m\,l/(N + m\,l) \qquad (40)$$

where $l\,\omega/k$ is the average rate of twist (rotational transform) of a field line relative to that of the helical field, running from 0 on the axis to 1 at the separatrix.

An obvious deduction from eq. 40 is that for any N the resonances get closer together as m increases, that is, as the separatrix is approached. Although small m may generally be more important, the contribution of large m will become significant near the separatrix. We conclude, therefore, that in no case would flux surfaces exist near the separatrix.

Since the spread of any resonance depends on $(r/R)^{N/2}$ we also expect small N to be the most dangerous. The $N = 1$ resonance cannot occur for a rotational transform less than $l/(l+1)$ times that of the separatrix; similarly the $N = 2$ resonance cannot occur for less than $l/(l+2)$ of the separatrix transform. Consequently, we expect the breakup of the surfaces to occur at values of rotational transform bearing a simple ratio to that of the separatrix.

The other type of perturbation which might exist would be due simply to random field errors. Using eq. 33, we can easily find the Brownian diffusion coefficient, if the spectrum of these irregularities is known. Those with periods longer than the helical period cannot resonate, those with periods comparable to the helical periods should be small compared to the toroidal corrections. Let us consider then errors induced by very high Fourier n components, for example, by a set of randomly placed magnets around the torus.

These will produce localized fields

$$B_{rj} = H_0 \frac{\Delta r_j}{(\pi\,\alpha_j)^{\frac{1}{2}}} \exp \frac{-(\varphi - \varphi_{0j})^2}{\alpha_j} \cos(\theta - \theta_{0j}) \qquad (41)$$

where φ_0, the position of the magnet, and θ_{0j}, its orientation are random variables. We suppose that α_j, the width, is fairly small, that is, $\omega\,\alpha^{\frac{1}{2}} \ll 1$. This kind of magnetic field irregularity in fact, can be described by the "white noise" correlation function

$$\left\langle \frac{\delta B_r}{H_0}(\varphi)\,\frac{\delta B_r}{H_0}(\varphi') \right\rangle = \sigma^2\,\delta(\varphi - \varphi') \qquad (42)$$

where $\sigma^2 = 2(\Delta r)^2 N$ and N is the number of such irregularities per unit length. The direct substitution of eq. 42 into eq. 33 gives us

$$\frac{\partial f}{\partial z} = \pi \frac{\partial}{\partial r}\,\sigma^2\,\frac{\partial f}{\partial r} \qquad (43)$$

Conclusion

We have noted that the equation for magnetic flux lines may be written in the form of a Liouville equation. If we Fourier-analyze the field perturbations in the form

$$\varepsilon\,a_{n,m}\,e^{i(n\,\Omega t + m\,\theta)} \qquad (44)$$

then the equation can be solved approximately as in quasi-linear theory. We find that for a single perturbation the disturbance of magnetic lines is strongly localized around flux surfaces on which the rotational transform $\omega = \mathrm{d}\,\theta/\mathrm{d}t$ is resonant, that is, where $n\Omega = m\,\omega$. In fact, away from resonance the distortion of the surfaces goes as ε while at resonance it goes as $\varepsilon^{\frac{1}{2}}$. Passing to the case of several resonances we find that if the distortions produced by the individual resonances are not strong enough to overlap with the adjoining resonances then the disturbances remain localized. If such overlap does take place then a Brownian motion of flux lines and rapid destruction of surfaces results. For a torus with helical windings we find that the strength and density of resonances is low at the magnetic axis and very high near the separatrix so that one may expect to find good flux surfaces in some region close to the magnetic axis but not near the separatrix. The rotational transform on the limiting flux surface would be expected to bear a simple ratio to the transform of the original, unperturbed separatrix, as given by eq. 40.

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