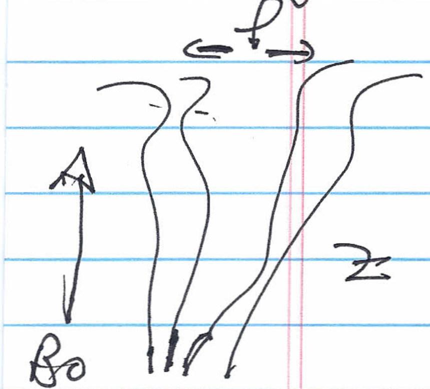
Transport - Stochastic Fields III

Misc

→ Can describe field line separation via spectrum of magnetic perturbations

→ approach follows that of Richardson

now $\ell \Leftrightarrow$ perp distance of lines



$$\frac{dr}{dz} \simeq \frac{d\ell}{dz} \simeq b_r$$

but $b_r \sim b_r(\ell)$ for Alfvenic turbulence, so

$$b_r(\ell) \sim \frac{\epsilon^{1/3}}{V_{A0}} \ell^{1/3}$$

$$\Rightarrow \quad \frac{d\ell}{dz} \sim \frac{\epsilon^{1/3}}{V_{A0}} \ell^{1/3}$$

$$\frac{d\ell}{\ell^{1/3}} \sim \frac{\epsilon^{1/3}}{V_{A0}} dz$$

$\therefore$

$$\Rightarrow \quad l^{2/3} \sim \frac{\epsilon^{1/3}}{V_{A0}} z$$

$$\therefore \quad l_\perp^3 \sim \frac{\epsilon}{V_{A0}^3} z^3$$

and can define correlation length

$$\langle k_\perp^2 l_\perp^2 \rangle \sim 1 \quad \Rightarrow \quad \frac{\epsilon k_\perp^2 l_c^3}{V_{A0}^3} \sim 1$$

$$\Rightarrow \boxed{l_c \sim \left(\epsilon k_\perp^2 / V_{A,0}^3\right)^{-1/3}}$$

can compare to:

$$l_c = \left(k_\perp^2 D_M / L_s^2\right)^{-1/3}.$$

→ Channels

– particles
- why speak of heat transport
- → test particle diffusivity
- why not particle transport?

Note: The 'point' of stochastic fields is

$\chi_\perp \sim v_{the} D_M$

↳ electron speed.

Useful to:

→ consider kinetic eqn.

→ distinguish between:

a) self-consistent case

$\tilde{B}$ produced by $\tilde{J}_\parallel$ in plasma

(i.e. EM instabilities – Ampere's Law)

b) $\tilde{B}$ produced by external means (coil)

Now,

$$\frac{\partial f}{\partial t} + v_\parallel \nabla_{\parallel 0} f + v_\parallel \frac{\delta B}{B_0} \cdot \nabla_\perp f + \cdots = 0$$

then integrating,

$$\frac{\partial \langle n_e \rangle}{\partial t} + \underline{\nabla}_\perp \cdot \left\langle \frac{\delta B_r}{B_0} \frac{\tilde{J}_{\parallel e}}{(-n_0|e|)} \right\rangle + \dots = 0$$

but

$$\nabla_\perp^2 A_\parallel = -\frac{4\pi}{c} \tilde{J}_\parallel$$

$$= -\frac{4\pi}{c} \left(\tilde{J}_{\parallel e} + \tilde{J}_{\parallel i} \right)$$

$$\tilde{J}_{\parallel e} = -\frac{c}{4\pi} \nabla_\perp^2 A - \tilde{J}_{\parallel i}$$

$$= -\frac{c}{4\pi} \nabla_\perp^2 A_\parallel - n_0|e| \underset{\substack{\uparrow \\ \underline{\underline{\text{ion flow}}}}}{\hat{V}_{\parallel i}}$$

so,

$$\frac{\partial \langle n_e \rangle}{\partial t} + \nabla_\perp \cdot \left\langle \frac{\delta B_r}{B_0} \left(-\frac{c}{4\pi} \nabla_\perp^2 \hat{A}_\parallel - n_0|e| \hat{V}_{\parallel i} \right) \right\rangle + \dots$$

$$= 0$$

$$\delta B_r = \nabla_\theta \hat{A}_\parallel$$

$\Rightarrow$

$$\frac{\partial\langle n_e\rangle}{\partial t} + \frac{\partial}{\partial r}\left\langle \nabla_\theta A_\parallel\left(-\frac{c}{4\pi}\nabla_\perp^2 \hat{A}_\parallel - n_0|e|\, v_\parallel\right)\right\rangle$$

$$+ \qquad = 0$$

so

$$\frac{\partial\langle n_e\rangle}{\partial t} = \frac{c}{4\pi}\frac{\partial}{\partial r}\overset{\text{①}}{\left\langle(\nabla_\theta \hat{A}_\parallel)(\nabla_\perp^2 \hat{A}_\parallel)\right\rangle}$$

$$+\, n_0|e|\overset{\text{②}}{\left\langle \delta B_r\, \hat{v}_{\parallel e}\right\rangle}$$

① : Taylor identity calculation !,

$$\left\langle(\nabla_\theta \hat{A}_\parallel)(\partial_r^2 A_\parallel + \underset{\text{odd}}{\cancel{\nabla_\theta^2 \hat{A}_\parallel}})\right\rangle$$

$$= \left\langle(\nabla_\theta A_\parallel)\partial_r^2 A_\parallel\right\rangle$$

$$= \left\langle \partial_r(\nabla_\theta A_\parallel)(\partial_r A_\parallel)\right\rangle - \overset{\text{odd}}{\cancel{\left\langle(\nabla_\theta \partial_r A_\parallel)(\partial_r A_\parallel)\right\rangle}}$$

$$= \partial_r\left\langle(\nabla_\theta \hat{A}_\parallel)(\partial_r A_\parallel)\right\rangle$$

$$= -\partial_r\left\langle(\delta B_r)(\delta B_\theta)\right\rangle$$

magnetic stress.

$$\frac{\partial \langle n_e \rangle}{\partial t} = -\frac{c}{4\pi}\, \partial_r \langle (\delta B_r)(\delta B_\theta) \rangle$$

↳ magnetic stress

$$+\, n_0 |e| \langle \delta B_r \hat{V}_{\parallel i} \rangle$$

↳ magnetic "flutter" of parallel ion flow.

Points:

- no explicit dependence on small electron inertia; i.e. $1/m_e$
- relation of magnetic stress to electron particle transport due stochastic fields

  i.e. ⇒ suggests effect on zonal flows, as zonal flows are fundamentally charge transport effect.

- fluctuating parallel ion flows + magnetic tilt ⇒ change electron density.

N.B. For external field, key is to calculate

- δB_r in plasma
- $\tilde{v}_{\parallel e}$ in plasma.

→ Re: Zonal Flows

Point: Magnetic wandering impacts zonal flows via charge

Recall: - fluctuations → DW model i.e. H-M, etc

- zonal mode: charge separation, radially

$\underline{\nabla} \cdot \underline{J} = 0$

usual e.s.; $\nabla_r J_{r\,pol} = 0$

⇒

$$\underbrace{\frac{\partial}{\partial t} \nabla_\perp^2 \phi}_{\substack{\text{linear} \\ \text{pol. drift}}} + \underbrace{\underline{V} \cdot \underline{\nabla}\, \nabla^2 \phi}_{\text{NL pol. drift}} = 0$$

so, for $\langle\ \rangle$ → zonal symmetry

[redacted]

$$\frac{\partial}{\partial t}\langle \nabla_r^2\phi\rangle + \partial_r\langle \tilde{V}_r \nabla_r^2\tilde{\phi}\rangle \dots = 0$$

↓ polarization charge Q_{pol}.

↓
- flux of polarization charge (i.e. net difference ions, electrons)
- relate to Reynolds stress by Taylor identity
- compute via modulation

so, can view:

$$\partial_t\langle Q_{pol}\rangle + \partial_r\langle \tilde{V}_r Q_{pol}\rangle = 0$$

$$i \to \quad \Gamma_{charge} = \left(\langle \tilde{U}_r \tilde{n}_{i_{GC}}\rangle + \langle \tilde{U}_r \tilde{Q}_{pol}\rangle\right)|e| - [redacted]\left(\langle \tilde{U}_r \tilde{n}_{e_{GC}}\rangle\right)|e|$$

$$\hat{n}_i = \hat{n}_e \quad \Rightarrow \quad \Gamma_{\text{charge}} = 0 \qquad \checkmark.$$

But magnetic perturbations induce new means charge transport

$$\Rightarrow \underline{\nabla}\cdot\underline{J} = 0$$

$$\nabla_\perp \cdot \underline{J}_{pol} + \nabla_\parallel \overline{J}_\parallel = 0$$

$$\nabla_\parallel = \nabla_\parallel^{(0)} + \frac{\delta \underline{B}}{B_0}\cdot\nabla_\perp$$

$\therefore$

$$\frac{\partial}{\partial t}\langle \nabla^2\phi\rangle + \langle \underline{\tilde{v}}\cdot\nabla\nabla^2\tilde{\phi}\rangle = \langle \underline{\tilde{B}}\cdot\nabla\tilde{J}_\parallel\rangle$$

obvious from RD MHD

ion pol. charge transport ⇒ advection

current flow along tilted lines

(n.b. current: electrons − ions)

so

$$\partial_t \langle \nabla^2 \phi \rangle = -\partial_r \left\{ \langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle - \langle \tilde{B}_r \tilde{J}_\parallel \rangle \right\}$$

so current flow along tilted lines acts to affect charge balance

⇒ enters effect on Z.F.

Note:

- sign not clear apriori
- for (~) Alfvenic fluctuations ⇒ damps
  i.e. acts cancel e.s.
- obviously related to electron particle transport.

⇝ What Happens for $Ku \gtrsim 1$?

— Recall:

~ Stoch Fields

$$Ku \sim \frac{l_{ac}}{\Delta_\perp}\, \delta B/B_0 \sim \frac{1/\Delta_\perp \;\; \delta B/B_0}{\Delta|k_\parallel|}$$

$$\sim 1/l_{NL} \,/\, 1/l_{ac} \sim l_{ac}/l_{NL}$$

ratio of autocorrelation to NL mixing length

Large $Ku \Rightarrow$ NL scatt. processes control time/sp. scale.

~ flow

$$Ku \sim \tilde{\tau}_{ac}\, \tilde{V}/\Delta \sim 1/\tau_{NL} \,/\, 1/\tau_{ac} \sim \tau_{ac}/\tau_{NL}$$

ion

i.e., collisional DW's:

$$\tau_{ac} \sim (\Delta|\chi_\parallel k_\parallel^2|)^{-1}$$

$$Ku \sim 1/\tilde{\tau}_c\, (\Delta|\chi_\parallel k_\parallel^2|)$$

2D GC Plasma — Simple / Compelling Example.

so Recall:

$$- D_\perp \sim \int_0^\infty d\tau \, \langle \tilde{V}(0) \tilde{V}(\tau) \rangle$$

Diffusion coeff as integral of correlation (i.e. time history ⇔ resp)

$$\sim \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 \underset{\substack{\uparrow \\ \text{Memory fctn.}}}{R(\tau)}$$

{ Diffusion coefficient as integral of correlation function

$$R(\tau) = e^{+i(\omega - k_\parallel v_\parallel)\tau} \, e^{-\tau/\tau_\perp}$$

From upo

Ku >> 1 limit corresponds to:

- → $k_\parallel \to 0$
- → $\omega \to 0$

} ⇒ { time integral controlled by <u>nonlinear</u> scattering, not wave packet dispers

⇒ 2D GC Plasma / Fluid:

i.e.

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} \rho = D_0 \nabla^2 \rho$$

$$\nabla^2 \phi = -4\pi \rho$$

(Taylor & McNamara)

→ 2D Fluid

→ GC Plasma.

Then generally,

$$D_\perp \cong \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 e^{-i\underline{k}\cdot\underline{r}_0} e^{+i\underline{k}\cdot\underline{r}(-\tau)}$$

but $\underline{r}(-\tau) = \underline{r}_0 + \underset{\downarrow}{\underline{\delta r}(-\tau)}$

only evolution off of $\underline{r}_0$

δr → stochastic

→ need ensemble average

$$D_\perp \cong \int_0^\infty \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 \left\langle e^{i\underline{k}\cdot\underline{\delta r}(-\tau)}\right\rangle d\tau$$

$$\cong \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 \underset{\downarrow}{e^{-k_\perp^2 D_\perp \tau}} d\tau$$

turbulence/scatt. itself controls correl^n. time.

i.e.

$$\left\langle e^{i\underline{k}\cdot\underline{\delta r}(-\tau)}\right\rangle \cong \left\langle \left(1 + \cancel{i\underline{k}\cdot\underline{\delta r}(-\tau)} - \frac{(\underline{k}\cdot\delta r(\tau))^2}{2} + \dots\right)\right\rangle$$

$$\cong \left\langle \left(1 - \frac{(k_\perp\cdot\delta r)_\perp^2}{2}\right)\right\rangle$$

$$\cong \left\langle \left(1 - k_\perp^2 D_\perp \tau - \right)\right\rangle$$

$$\cong e^{-k_\perp^2 D_\perp \tau}$$

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$$\underset{\uparrow}{D_\perp} = \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 e^{-k_\perp^2 D_\perp \tau} d\tau$$

$$= \sum_k |\tilde{V}_k|^2 \frac{1}{\underset{\uparrow}{k_\perp^2 D_\perp}} \Bigg\}$$

can immediately note:

~~[illegible]~~

— $D_\perp$ controls τ_c

— conservation Θ

$\partial_t \int \Theta \, d^2x \to 0 \Rightarrow$ slow τ_c at large scale

N.B. → note recursive structure of diffusivity!

→ τ_c in integral set by scattering $\sim 1/k_\perp^2 D$, $k_\perp^2$ from cons.

⇒ → $\underline{V} \sim \frac{c}{B} \underline{E} \times \hat{z}$ → $D_\perp \sim 1/B_0$ ⇒ Bohm.

$$\boxed{(D_\perp)^2 \cong \sum_k |\tilde{V}_k|^2 / k_\perp^2}$$

→ recursive definition

but 2D, assuming symmetric spectrum.

$$D_\perp^2 \cong \int dk_\perp \cancel{k_\perp} |\tilde{V}_k|^2 / k_\perp^{\cancel{2}}$$

$\Rightarrow$

$$D_\perp^2 \cong \frac{c^2}{B_0^2}\int dk_\perp \frac{|E_k|^2}{k_\perp}$$

$$\boxed{D_\perp \sim \frac{c}{\underset{\overset{\uparrow}{\sim 1/B_0}}{B_0}}\left(\int dk_\perp \frac{|E_k|^2}{k_\perp}\right)^{1/2}} \Rightarrow \text{Diffusivity}$$

Now, can explore different spectra:

~ thermal equilibrium: (a la TPM)
→ thermal

$$|E_k|^2 = \frac{4\pi}{\ell} k_B T \Big/ \left(1+k^2\lambda_D^2\right)$$

e/ℓ → charge/length. ↪ Debye screening

$$= \frac{4\pi}{e}\left(\frac{e}{\ell}\right) k_B T \Big/ 1+k_\perp^2\lambda_D^2$$

$$D_\perp \sim \frac{c}{B_0}\left(\int_{1/L}^{1/\lambda_D} dk_\perp \left[\frac{4\pi}{e}\left(\frac{e}{\ell}\right) k_B T \Big/ \left(\left(1+k_\perp^2\lambda_D^2\right)k_\perp\right)\right]\right)$$

$$\sim \frac{c k_B T}{eB}\left[(n\lambda^2)^{-1}\ln(L_0/\lambda)\right]^{1/2}$$

so

$$D_\perp \sim D_B \left[(n \lambda_D^2)^{-1} \ln(L_0/\lambda) \right]^{1/2}$$

- recovers basic Bohm scaling, even from thermal flctns
- scales (weakly) with L_0 ⇒ not local, or intensive

⇒ simple example of "non-locality"

- "non-locality" appears from "slow mode" i.e. $1/\tau_c \to 0 \sim k_\perp^2 D_\perp$ as $k_\perp^2 \to 0$

- if shear flow:

$$R(\omega) = \int_0^\infty d\tau \, \exp\left[i(\omega - k_0 V_0)\tau - \tau/\tau_c \right]$$

Interesting to note:

- can consider diffusion due random array charges ("rods")

For spectrum:

$$\underline{\nabla}\cdot\underline{E} = 4\pi\rho$$

$$= \frac{4\pi}{\ell}\sum_i q_i\,\delta(\underline{x}-\underline{x}_i)$$

$$i\underline{k}\cdot\underline{E}_k = \left(\frac{4\pi}{\ell}\right)\sum_i q_i\, e^{-i\underline{k}\cdot x_i}$$

symmetric distribution $\Rightarrow$ $\left\{\begin{array}{l}\text{random array}\\ \text{discrete charges}\end{array}\right.$ $\downarrow$

$$|E_k|^2 = \frac{1}{k_\perp^2}\left(\frac{4\pi}{\ell}\right)^2 \left\langle \sum_{i,j} q_i q_j\, e^{i k\cdot(x_j - x_i)} \right\rangle$$

$$= \frac{16\pi^2 n q^2}{\underbrace{k_\perp^2}_{\longrightarrow\ 1/k_\perp^2}\ell^2}$$

$$D_\perp^2 \sim \frac{c^2}{B_0^2}\int dk_\perp \frac{16\pi^2 n q^2}{k_\perp^2 \ell^2 k_\perp^2}\, k_\perp$$

$$\sim \frac{c^2}{B_0^2}\int_{k_{min}}^{k_{max}} \frac{16\pi^2 n q^2}{k_\perp^3 \ell^2} \sim \frac{c^2}{B_0^2}\left(\frac{16\pi^2 n q^2}{\ell^2}\right)\frac{1}{k_{\perp\,min}^2}$$

$$k_{\perp \min} \sim 1/\underset{\substack{\uparrow \\ \text{system size}}}{L_0}$$

$$\boxed{D_\perp \sim \frac{c}{B_0}\,4\pi\left(\frac{n\xi^2}{\rho}\right)^{1/2} L_0}$$

$\uparrow$

strong dependence on
system size.