Introduction to Intermittency

Lecture: Prof. Pat Diamond, Notes: Jiacong Li

1 Overview

Intermittency denotes the rare occurrence of very spiky events. These events are patchy and bursty. In terms of statistics, in intermittent systems, higher order moments do not converge, suggesting significant departure from Gaussian statistics. In fact, intermittency emerges from multiplicative processes, such as chemical reaction, and particle distribution with a random potential. Three examples of intermittent systems are discussed.

- Multiplicative noise, which is often seen in Vlasov turbulence and MHD turbulence, leads to log-normal distribution of fluctuations, which suggests the correlation between intermittency and multiplicative processes.
- Particle occupation density in a random media shows that occupation density is concentrated in space. Moments are controlled by both the order and the most probable potential. The resulting large kurtosis, which departs from order 1, invalidates most existing closure theories.
- Reaction-diffusion systems with random reactions exhibit concentration in the diffusion trajectory space. In particular, diffusion trajectories are controlled by the random reaction, as the latter is in a "stronger" state of randomness.

Given the patchy and bursty nature of intermittent systems, interest is in the geometrical dimensions of these systems. Further, the fraction of active volume in Navier-Stokes turbulence leads to inhomogeneous dissipation and modified dissipation scale, and thus induces memory of larger scales. This will be left for the following notes on intermittency.

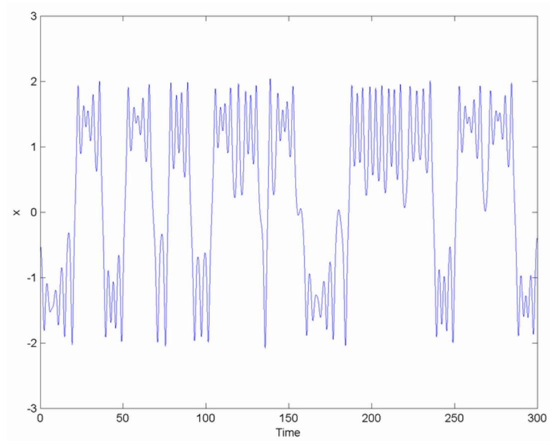

Figure 1: Intermittent jumping between two potential wells in the driven Duffing oscillator.

2 What is intermittency?

Some events have probability distribution function (PDF) that is concentrated in widely spaced patches or intervals where the local PDF is much larger than the mean PDF, i.e. $f \gg \overline{f}$. For example, jumping between two potential wells in the driven Duffing osscillator (Fig.1). Other examples of intermittent systems include: flares in the solar corona, magnetic sub-storms in the magnetosphere, and edge-localized modes (ELMs) in the edge of H-mode tokamak plasmas. The appearance of such patchy and bursty structures is called intermittency. Percolation theory aims at developing models of transport in intermittent systems. Thus, we first discuss intermittency.

In terms of states of randomness, intermittency comes along with slow or wild randomness, not mild. In order to deal with real world turbulence, i.e. financial modeling, Mandelbrot broadens the scope of statistical models by introducing three main states of randomness: mild, slow, and wild (Fig.2-4). Mild randomness deals with Gaussian distribution, i.e. systems with well defined mean and variance (2nd order moment). Slow randomness denotes the state of randomness with growing higher order moments, e.g. the intermittent system discussed here as well as other systems with lognormal distribution. Wild randomness is charactered by the absence of convergence for even the lowest order moment (i.e. variance), e.g. power law distribution. Percolation threshold may be related to the change in state of randomness from mild to wild.

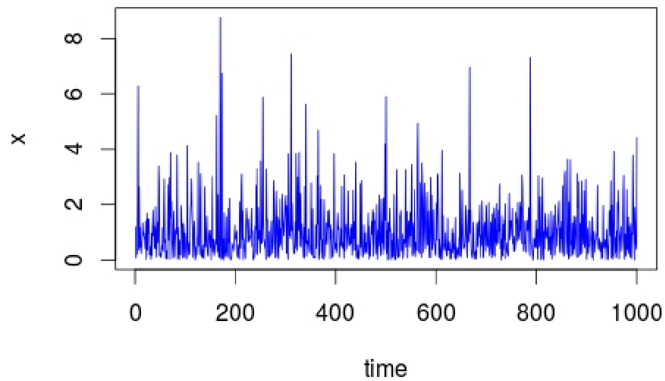

Figure 2: Random draws from an exponential distribution with mean=1. (Mild randomness)

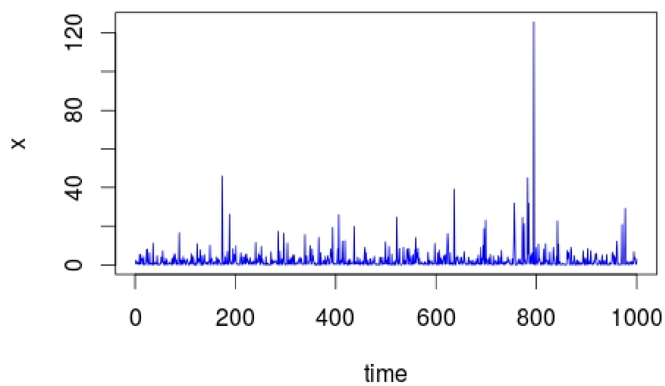

Figure 3: Random draws from a lognormal distribution with mean=1. (Slow randomness)

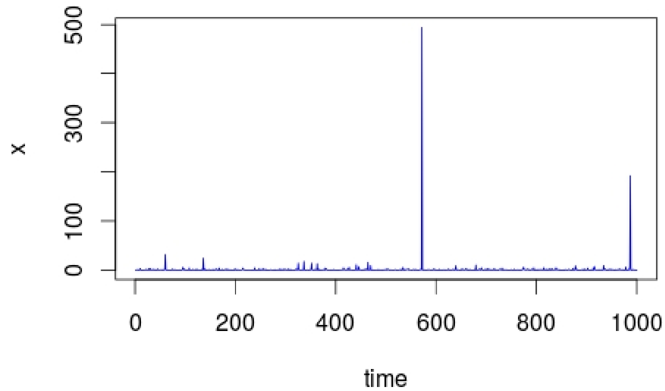

Figure 4: Random draws from a Pareto distribution with mean=1 and α=1.5. (Wild randomness)

3 How to identify and deal with intermittency?

Characters of intermittency imply departure from familiar world of Central Limit Theorem (CLT) and Law of Large Numbers (LLN). This means a departure from Gaussian statistics, i.e. smooth distribution. First, we'll recall the basics of CLT.

3.1 Additive Process: Recall Central Limit Theorem

Let $x_1, x_2, ...$ be a sequence of *independent* and *identically distributed* random variables, each with mean μ, and standard deviation σ, so:

$$\overline{x_i} = \mu,$$

$$\langle (x_i - \overline{x_i})^2 \rangle = \sigma^2,$$

then, the addition of these random variables, i.e.

$$\frac{x_1 + x_2 + ... + x_N - N\mu}{\sqrt{N}\sigma}$$

is distributed as a Gaussian (Fig.5). This is to say, sum of a sequence of random variables converges to Gaussian distribution of width $\sqrt{N}\sigma$, if these variables have:

- No correlations;
- Same distribution, i.e. there are no "special" steps;
- Well defined σ^2, i.e. there are no fat tails in the distributions.

3.2 Multiplicative Process with 2-Point Distributed Components

Define

$$X = \prod_{i=1}^{N} x_i = x_1 x_2 ... x_j ... x_N,$$

with

$$x_j = \begin{cases} 0, & p = 1/2; \\ 2, & p = 1/2. \end{cases}$$

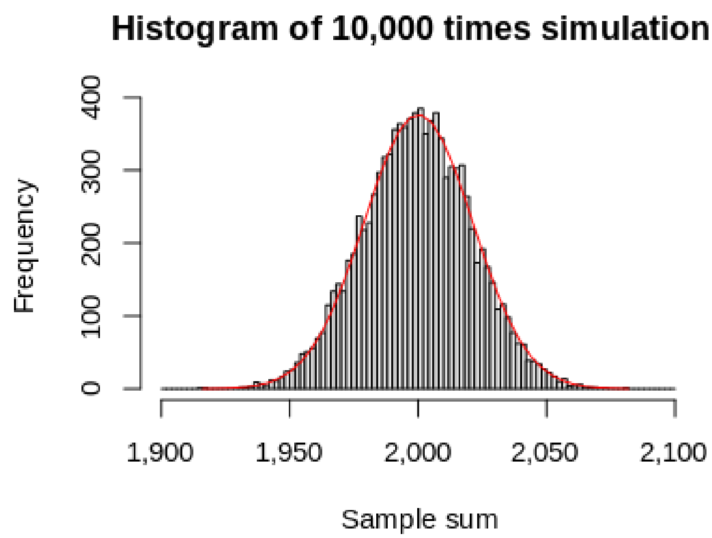

Figure 5: Histogram of 10,000 times of Monte Carlo simulation. In each simulation, sum of 1,000 samples are recorded. For each sample, one of {1, 2, 3} is randomly chosen.

We are seeking to obtain $\langle X \rangle$ and $\langle X^2 \rangle$. An observation is that $X = 0$ unless all x_i's are 2. Then, we have

$$X = \begin{cases} 2^N, & p = 2^{-N}; \\ 0, & p = 1 - 2^{-N}. \end{cases}$$

So from this probability distribution of X,

$$\langle X \rangle = \sum_j X_j p(X_j) = 1, \tag{1}$$

$$\langle X^2 \rangle = \sum_j X_j^2 p(X_j) = 2^N, \tag{2}$$

where the index j corresponds to realizations of X. This result means that as the number of multiplicative components $N \to \infty$, *the variance blows up while the mean stays constant.* This departs from the CLT, which requires variance to be convergent as the number of additive components increases.

More generally, higher order moments of the above distribution grow faster as the the order goes up. Explicitly,

$$\langle X^p \rangle = \sum_j X_j^p p(X_j) = 2^{(p-1)N}. \tag{3}$$

The growth rate with N (like time) is

$$\gamma_p = \frac{\log_2 \langle X_p \rangle}{N} = p - 1. \tag{4}$$

Here, N is surrogate for time or number of time steps. Note that higher order moments grow faster.

The increasing growth of higher moments is the signature of an intermittent process. Thus, X is an intermittent random quantity. It is also import to note that intermittent random quantity is the result of *multiplying*, not adding, many random numbers.

3.3 Multiplicative Process with Log-Normal Distributed Components

Consider a sequence of random numbers distributed around unity, $\{x_i\}$. Define

$$X = \prod_{i=1}^{N} x_i.$$

This multiplicative process can be transformed into a additive process by taking the logarithm, i.e.

$$\ln X = \sum_{i=1}^{N} \ln x_i.$$

Now, we can apply CLT to this sum of log's. As $N \to \infty$, $\ln X$ is distributed as

$$p(\ln X) \sim \exp\left[-(\ln X)^2/N\eta^2\right]. \tag{5}$$

with $\ln X \sim \sqrt{N}\eta$, where η is a Gaussian random quantity with unit dispersion. So $X \sim \exp(\sqrt{N}\eta)$. This means X has a log-normal distribution:

$$p(X) = \frac{1}{X\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln X - \mu)^2}{2\sigma^2}\right], \tag{6}$$

as shown in Fig.6.

As N goes up, magnitude of X depends on the sign of η. Since η is a Gaussian random quantity with unit dispersion, its value lies within the range of one standard deviation, $(-\sigma, \sigma)$. So if $\eta < 0$, X is exponentially small, as N increases. More interestingly, when $\eta > 0$, $X \sim \exp(\sqrt{N}\sigma)$ so X is large. This implies a very spiky distribution, with probability concentrated in place.

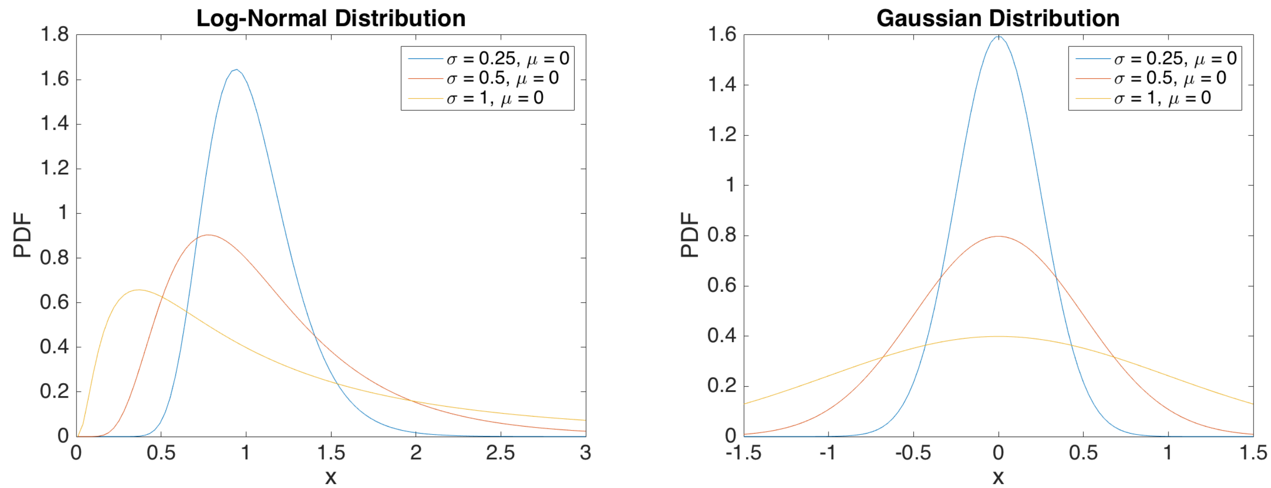

Figure 6: PDF of log-normal and Gaussian distributions with various scale parameters σ.

With $\eta > 0$, the mean value of X is

$$\langle X \rangle = \int X(\eta)p(\eta)d\eta = \int \exp(\sqrt{N}\eta)exp(-\eta^2/2\sigma^2) \sim \exp(N\sigma^2/2). \tag{7}$$

Thus, the average $\langle X \rangle$ grows exponentially with N. Moreover, higher moments also grow exponentially, since

$$\langle X^p \rangle \sim \exp[p^2 N\sigma^2/2]. \tag{8}$$

The grow rate is

$$\gamma_p \sim \lim_{N\to\infty} \frac{\ln\langle X^p \rangle}{N} = \frac{p^2\sigma^2}{2}. \tag{9}$$

Note that $\gamma_p \sim p^2$.

To summarize, log-normal distribution is a prototype distribution for multiplicative intermittent random variables. The rapid growth of higher moments with N is a symptom of *concentration*. By log-normal distribution, we mean $\ln X \sim \sqrt{N}\eta$ and thus $X \sim \exp(\sqrt{N}\eta)$, with η is a Gaussian random variable. For log-normal distribution, moments of all orders grow exponentially as N goes up, e.g. $\langle X \rangle \sim \exp(N\sigma^2/2)$.

4 Examples of Intermittent Systems

Three intermittent systems are introduced as examples: evolution with multiplicative noise, the particle distribution in random media, and reaction-diffusion systems with random reaction.

4.1 Evolution with Multiplicative Noise

Multiplicative random quantities arise in evolutionary problems. For example, in Vlasov plasmas with evolution of phase space distribution with a random electric field, i.e.

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q\tilde{\mathbf{E}}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,$$

and various MHD phenomena whose dynamics are governed by

$$\frac{\partial \mathbf{B}}{\partial t} \sim \nabla \times (\tilde{\mathbf{v}} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

This induction equation can be treated as a linear stochastic PDE.

First, consider a simple comparison between evolution with additive noise

$$\frac{d\psi}{dt} = \varepsilon(t) \tag{10}$$

and with multiplicative noise

$$\frac{d\psi}{dt} = \varepsilon(t)\psi, \tag{11}$$

where $\varepsilon(t)$ is noise with Gaussian distribution with dispersion σ^2 and small correlation time, i.e. $\langle \varepsilon(t)\varepsilon(t') \rangle \sim \delta(t - t')$, which means each single time step ($\sim \tau_c$) is statistically independent from others.

The additive noise clearly results in a diffusion process, since

$$\langle \psi^2 \rangle \sim \int_0^t \varepsilon(t_1) dt_1 \int_0^t \varepsilon(t_2) dt_2 \sim \langle \varepsilon^2 \rangle \tau_c t,$$

with τ as the correlation time of the noise. In detail, assume ε re-sets after very time step τ_c, so

$$\psi = \int_0^t \varepsilon(t')dt' = \int_0^{\tau_c} \varepsilon(t')dt' + \int_{\tau_c}^{2\tau_c} \varepsilon(t')dt' + ... \tag{12}$$

Using CLT, we can obtain a random variable η with Gaussion distribution with unit dispersion ($\langle \eta^2 \rangle \sim 1$) such that

$$\psi = \langle \varepsilon \rangle t + \tau_c \sigma \left(\frac{t}{\tau} \right)^{1/2} \eta, \tag{13}$$

with σ is the standard deviation of ε distribution. Taking $\langle \varepsilon \rangle \sim 0$ for ε Gaussian, the diffusion given in the beginning of this paragraph is retrieved, which is

$$\langle \psi^2 \rangle \sim \tau_c \sigma^2 t. \tag{14}$$

The evolution with multiplicative noise requires *nonlinear* models. Here, by using the logarithm, a *linear* stochastic PDE is revealed as a model of *nonlinear* PDE:

$$\frac{d}{dt}\ln\psi = \varepsilon(t), \tag{15}$$

with

$$\psi(t) \sim \psi_0 \exp\left[\int_0^t \varepsilon(s)ds\right] \sim \prod_{n=0}^{t/\tau_c} \exp\int_{n\tau_c}^{(n+1)\tau_c} \varepsilon(s)ds. \tag{16}$$

Note here, the random process ψ is the multiplication of a sequence of uncorrelated random variables. Thus, $\ln\psi$ is a process with additive noise and follows CLT:

$$\ln\psi \sim \langle\varepsilon\rangle t + \sigma\tau_c \left(\frac{t}{\tau_c}\right)^{1/2} \eta, \tag{17}$$

where η is distributed according to a Gaussian with unit dispersion. Using $\langle\varepsilon\rangle = 0$, we find ψ has a log-normal distribution

$$\psi \sim \exp\left[\sigma\tau_c \left(\frac{t}{\tau_c}\right)^{1/2} \eta\right]. \tag{18}$$

Therefore, ψ has all the properties that have been discussed in Sec.3.3. This means that ψ grows exponentially with time as $\exp\left(t^{1/2}\right)$. Thus, ψ is an example of an intermittent, random quantity.

One lesson learned for this intermittent system is the fluctuations grow in time as $\exp(t/\tau_c)^{1/2}$. This would invalidate most of the closure theories, which all assume small fluctuations. So we want to compare the above stochastic approach to usual closure calculation, i.e. quasilinear theory (QLT) or mean field theory (MFT).

By MFT, first separate fluctuations from the slowly varying mean profile

$$\psi = \langle\psi\rangle + \tilde{\psi}.$$

Here,

$$\tilde{\psi} \ll \langle\psi\rangle$$

is assumed. Thus, by matching the orders, we can obtain the evolution of mean profile and the fluctuation:

$$\frac{d\langle\psi\rangle}{dt} = \langle\varepsilon\tilde{\psi}\rangle, \tag{19}$$

$$\frac{d\tilde{\psi}}{dt} = \varepsilon\langle\psi\rangle. \tag{20}$$

From the fluctuation equation, linear response of $\tilde{\psi}$ is, in QLT:

$$\tilde{\psi} = \int \varepsilon \langle \psi \rangle dt. \tag{21}$$

Using this linear response, the evolution of mean profile is

$$\frac{d\langle \psi \rangle}{dt} \cong \langle \varepsilon^2 \rangle \tau_c \langle \psi \rangle, \tag{22}$$

for small τ_c. So the mean profile grows as

$$\langle \psi \rangle \sim \psi_0 \exp\left(\sigma^2 \tau_c t\right). \tag{23}$$

In comparison, if we take the result for ψ from the stochastic approach and take the average, we can obtain

$$\langle \psi \rangle \sim \int d\eta \exp\left[\sigma \tau_c (t/\tau_c)^{1/2} \eta\right] \exp(-\eta^2) \sim \exp\left(\sigma^2 \tau_c t\right), \tag{24}$$

which agrees with the MFT result. *But*, MFT *won't* get higher order moments right!

In conclusion, MFT gets the first moment (average) right, but if we want to reveal fundamental nature and structure of higher moments, we need to consider intermittency buried in multiplicative processes. Intermittency leads to occurrence of rare but intense peaks in the behavior of a random quantity. Such peaks are represented by higher order moments.

4.2 Particle Density Distribution in Random Media

In this section, we consider the example of the particle density distribution in a random media, i.e. a continuum of random quantities. Random media can be defined by a random potential: $U(x, \omega)$ at temperature T. ω enumerates realizations of the media. Fixed ω corresponds to a specific configuration of particle density distribution. Here, we consider the case that U is Gaussian distributed with variance σ^2.

Given a potential U, particles are distributed according to:

$$n = n_0 \exp[-U/k_b T], \tag{25}$$

where n is the occupation density of particles, and k_b is the Boltzmann constant. Thus, distribution n is non-Gaussian, as its dependence on U is nonlinear.

The PDF of concentration is

$$p\left(n\left(U\right)\right) = n(U)p(U) = n_0 \exp(-U/k_bT)\exp(-U^2/2\sigma^2). \tag{26}$$

So the most probable concentration is at

$$U_{\max}/\sigma \sim -\sigma/k_bT \tag{27}$$

with probability density

$$P_{\max} = n_0 \exp(-\sigma^2/k_b^2T^2). \tag{28}$$

Moreover, higher moments of density distribution are

$$\langle n\rangle = n_0 \exp(\sigma^2/2k_b^2T^2), \tag{29}$$

$$\langle n^2\rangle^{1/2} = n_0 \exp(\sigma^2/k_b^2T^2), \tag{30}$$

$$\langle n^p\rangle^{1/p} = n_0 \exp(p\sigma^2/2k_b^2T^2). \tag{31}$$

Note that $\langle n^2\rangle > \langle n\rangle^2$, which suggests a *fluctuation dominated system*. In fact, higher moments become even larger, e.g. $\langle n^4\rangle \gg \langle n^2\rangle^2$. Successive moments are not determined by the most probable density, at $U \sim \sigma/k_bT$ but by the density at $U \sim p^{1/2}\sigma/k_bT$. These behaviors are signatures of an intermittent distribution of density. In such intermittent systems, normal closure models fail because most of them assume kurtosis of order 1. Kurtosis is defined as

$$K(x) = \frac{E\{(x-\overline{x})^4\}}{\left(E\{(x-\overline{x})^2\}\right)^2}. \tag{32}$$

It is a measure of the "tailedness" of the probability distribution of a random variable. Large kurtosis that significantly departs from order 1 suggests a "fat" tail in the distribution function. Fat tail is another way to describe the rarely occurring, very spiky events in a intermittent system.

After all, why the interest in higher moments? Because "Progressive growth of statistical moments with order can be explained only by a much more pronounced dominance of rare intense peaks in the concentration distribution."

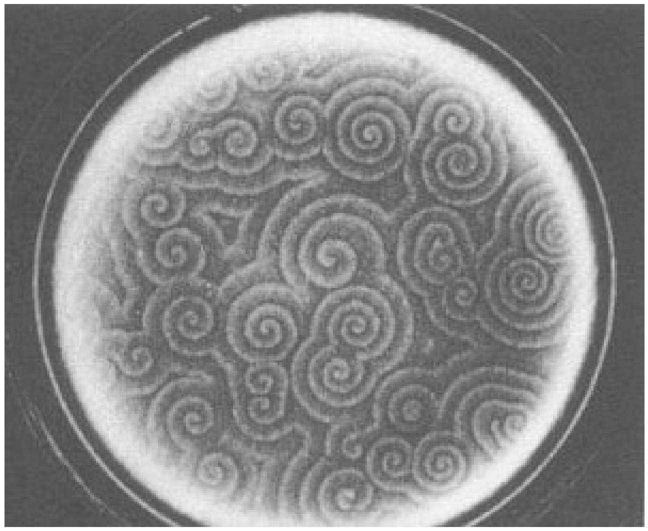

Figure 7: Spiral waves in colonies of the mold Dictyostelium discoideum. (BALL, 1994)

4.3 Reaction-Diffusion Process with Random Reaction

In this section, we consider the statistics and moments of linear scalar equations that contain both random growth (potential) and diffusion, i.e. stochastic reaction-diffusion processes with random reaction. While diffusion–an additive process–falls in the state of mild randomness, reaction is a multiplicative process and would lead to intermittency. Being a "stronger" state of randomness, the intermittent reaction dominates the reaction-diffusion system. Thus, this kind of systems demonstrate patchy and bursty events, such as spiral wave patterns in Fig.7.

The equation for this system is

$$\frac{\partial \psi}{\partial t} = D\nabla^2\psi + U(x,\omega)\psi, \tag{33}$$

with initial condition $\psi(x,0) = \psi_0$. Here, U denotes stochastic interaction and is distributed as Gaussian with correlation $\langle U(x)U(x')\rangle \to 0$ as $|x - x'| > l_c$.

To solve the equation, first rewrite the equation in terms of operators (as we do in quantum mechanics):

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= D\nabla^2\psi + U\psi \\ &= H\psi = (H_0 + U)\psi. \end{aligned} \tag{34}$$

Thus, we have

$$\psi = \exp\left(\int H dt\right) = \exp(H_0 t)\exp\left(\int U ds\right).$$

Thus, we can obtained the average $\langle\psi\rangle$ over the trajectories of H_0, which is

$$\langle\psi\rangle = \langle\exp\left(\int U ds\right)\rangle. \tag{35}$$

Therefore, the evolution of ψ can be obtained by the path integral:

$$\psi(x,t) = M_x\left[\exp\left(\int_0^t U(\varepsilon_s)ds\right)\psi_0(\varepsilon_s)\right], \tag{36}$$

where the operator $M_x \equiv$ average over trajectories of Brownian motion:

$$\varepsilon_s(x) = (2D)^{1/2}W_s(x) \tag{37}$$

which begins at ε_l at $s = 0$ and ends at x at $s = t$. This is similar to cumulants in resonance-broadening theory.

The message here is that dominant contributions are from that trajectory which encounters high value of potential, leading to concentration "points" in the trajectory space. Therefore, in the spirit of fastest descent for asymptotic approximation, the path integral is dominated by the height of maximum U.

Now, consider a region of radius R, such that $R \gg l$, then the number of cells in the R^3 ball is $(R/l)^3$. The maximum U requires $(R/l)^3 p(U_{\max}) \sim 1$. Therefore, for U with Gaussian distribution, $p(U_{\max}) \sim \exp(-U_{\max}^2/2\sigma^2)$, the maximum U is

$$U_{\max} \sim \left[6\sigma^2 \ln(R/l)\right]^{1/2}. \tag{38}$$

With R controlled by diffusion, i.e. $R/l \sim (Dt)^{1/2}/l \sim (Dt/l^2)^{1/2}$, the evolution of ψ is

$$\psi \sim \exp\left(\int U ds\right) \sim \exp\left\{t\left[3\sigma^2\ln(Dt/l^2)\right]^{1/2}\right\}. \tag{39}$$

This evolution is a *super-exponential* growth ($\sim \exp(t\ln t)$) in time, even for the typical (most probable) trajectory which has been considered here.

Note that even for $D = 0$, higher moments still grow super-exponentially. This is because for

$$\frac{\partial\psi}{\partial t} = U\psi, \tag{40}$$

we have

$$\psi = \exp(Ut)\psi_0. \tag{41}$$

Thus, following the process in Sec.3.3,

$$\langle \psi^p \rangle = \langle \exp(pUt) \rangle \psi_0^p = \psi_0^p \exp(p^2\sigma^2 t^2/2). \tag{42}$$

Finally, we can obtain the higher moments:

$$\langle \psi^p \rangle^{1/p} = \langle \exp(pUt) \rangle \psi_0^p = \psi_0 \exp(p\sigma^2 t^2/2). \tag{43}$$

Therefore, higher moments exhibit super-exponential growth and they grow faster than ψ itself! However, without diffusion fluctuations grow faster as $\psi \sim \exp(t^2)$, compared to $\psi \sim \exp(t \ln t)$ with diffusion. This means diffusion moderates the growth of fluctuations and the diffusion trajectories are controlled by the maximum reaction potential.

5 Summary and What's Next?

So far, we have seen the characteristics of intermittent systems with three examples: evolution with multiplicative noise, particle distribution in random media, and reaction-diffusion with random reaction. All these systems demonstrate growing higher order moments and concentration. In particular, the lessons learned here are:

- Intermittency is the result of multiplicative processes, not additive processes.
- Growing higher order moments is the signature of intermittency. Higher order moments grow as $\sim \exp(p)$. The resulting large kurtosis invalidates most existing closure theories.
- Intermittency is a stronger randomness than diffusion. Diffusion trajectories are modified by intermittent reaction processes in a reaction-diffusion system with random reactions.

We have seen one character of intermittency is spatial concentration, which may lead to fractal geometries. In particular, concentration is related to vortex tube stretching, which may explain the higher order moments in Navier-Stokes turbulence (NST). What is the active volume resulting from the concentration? As of NST with concentration, fractal geometry will lead to nonuniform dissipation, raising the problem of dissipative structure dimension as well as reduction in the fraction of active volume. These will come up in the following lectures on intermittency.