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REVIEW ARTICLE

Sandpiles, silos and tokamak phenomenology: a brief review

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Abstract. Increasing attention is being given to the possibility that some key features of tokamak confinement physics may not be specific to the plasma state. In particular, there appear to be parallels between sandpiles and transport in tokamaks. Aspects of sandpile physics involving self-organized criticality (SOC) have been suggested by Carreras, Diamond, Newman and co-workers (for example, 1996 *Phys. Plasmas* **3** 1858, 2903 and 3745) as a paradigm for certain tokamak phenomena. However, the range of confinement physics displayed by sandpiles (in particular, by real experimental sandpiles) and also by related systems such as silos is substantially broader than so far considered for application to tokamak plasmas. It is reviewed in this paper, and additional candidate phenomena for sandpile modelling in tokamak physics, such as edge-localized modes and Berk–Breizman dynamics, are identified. The behaviour of sandpile-like systems is varied, and is sometimes sensitive to details in the experimental set-up or the theoretical model, so the role of SOC is not entirely clear. Recent theoretical progress in the modelling of experimental sandpiles is described, showing that the internal dynamics of a sandpile-type system may be governed by SOC even when this is not observed in the flow of matter from the system.

1. Introduction

It is no longer taken for granted that all the distinctive features of tokamak confinement physics are necessarily specific to the plasma state. Attention is now being given to the possibility that some of the observed phenomena may be generic to a broader class of physical systems that involve the confinement of macroscopic quantities of matter under non-equilibrium conditions. At present, a leading candidate paradigm is self-organized criticality (SOC) associated with certain idealized models for the dynamics of sandpiles [1, 2]. This has been applied to turbulent transport in tokamaks by Carreras, Diamond, Newman and co-workers [3–6], and to reconnection in astrophysical plasmas by Lu and Hamilton [7]. At the simplest level, there appear to be physical parallels between sandpiles and transport in tokamaks. The global properties of the latter may arise from the growth, saturation and nearest neighbour couplings of a discrete sequence of spatially localized modes centred on successive resonant surfaces. Sandpile dynamics (whether displaying SOC or otherwise) are typically modelled in broadly similar terms, involving a sequence of nodes at which sand is added until a critical gradient is reached locally, triggering a redistribution of sand to nearest-neighbours, followed by iteration. The experimental and theoretical literature relating to sandpiles (and to their near relatives, silos) is, in fact, very extensive, and describes a phenomenology far richer than has yet been considered for application to tokamak physics. Indeed, the sandpile and silo literature encompasses one of the broadest range of confinement physics issues outside fusion, and it is appropriate to investigate whether others among these issues may be generic to both fields. For example, we may be able to learn from the behaviour of real sandpiles which are actual physical confinement systems, as distinct from theoretical paradigms such as SOC.

The notion of self-organized criticality (SOC) was introduced by Bak *et al* [1], who showed that certain systems of simple cellular automata evolve to a critical state through a self-organization process. The state is termed critical because it has no characteristic length or time scales (or only very faint traces thereof); it is self-organized since it is completely insensitive to initial conditions. The simplest such systems are referred to as mathematical sandpiles. The state of the system is specified by an array of integers representing the height of the pile (number of sand grains) at each position. At each time step a grain of sand is added at a random position. If the local slope of the pile (the height difference between neighbouring positions) exceeds some critical value, sand is redistributed in such a way as to reduce the local slope to some value below the critical one. This may result in the critical gradient being exceeded at the neighbouring positions, which, in turn, leads to redistribution there. The process may then spread further, leading to an avalanche. Avalanches may involve loss of sand grains from the pile, or may involve only a rearrangement of the grains internal to the pile. The distribution of avalanches (and other events) in mathematical sandpiles has been investigated by Kadanoff *et al* [2], and is found typically to follow a simple scaling law (for example, a power law) over a range of several orders of magnitude in the mass of the sand involved in the event. The overall behaviour of the system turns out to be independent of the critical gradient and of the finer details of the algorithm determining the number of redistributed sand grains, provided that the redistribution overcompensates in the sense that a supercritical gradient is restored to some value below, rather than exactly at, criticality.

The status of results that are inferred from SOC models applied to tokamak plasmas is a topic of current debate within the fusion community. Fundamental questions include: the nature of the link between sandpile algorithms, which may appear to be heuristic, and analytical models that can be derived from the basic principles of plasma physics; by extension, the extent to which SOC models are, or can be, 'rigged' to display the phenomenology of choice, given that, perhaps unavoidably, some of the SOC models proposed so far in tokamak physics are not derived from first principles of plasma physics; and the degree of robustness, that is, invariance of the phenomenology against minor changes of the sandpile algorithm, which should be possessed by an SOC model of a given plasma process.

The first objective of this paper is to provide a brief but comprehensive review of potentially fusion-relevant aspects of the rapidly developing and cross-disciplinary field of sandpile and silo physics, so as to improve its accessibility to the fusion plasma community. We then revisit some of the tokamak phenomenology already identified in [3–7], highlighting some of the more striking experimental observations, which we believe could also be of interest beyond the tokamak community. Next, we explore additional tokamak phenomena, emphasizing those aspects that appear potentially suited to sandpile-type modelling. The final section of the paper describes a recent model for experimental sandpile observations that sheds light on the role of SOC in sandpile-like systems.

2. Brief review of sandpile physics

Let us turn first to real sandpiles [8–13] which, it is apparently accepted [8, 14–16], do not usually (with one notable exception [8, 13]) exhibit SOC avalanches. Experiments have been carried out using: slowly rotating drums partially filled with sand [9, 10]; conical sandpiles with grains fed to the apex [11, 12]; sandpiles fed by random sprinkling of particles over the active surface [9] and piles of rice grains [8, 13]. Tilting a rotating drum is equivalent, in some ways, to fuelling from below; clearly, some forms of sandpile fuelling are more random than others. Re-analysis in [15] of the data of [9–12] suggests the absence of SOC, while the occurrence or otherwise of SOC in [8] depends on the aspect ratio (length/diameter) of the rice grains. The type of experimental data obtained includes: amplitude and time separation of avalanches involving mass loss from the sandpile, for example, figure 1 of [9]; time evolution of sandpile mass, for example, figure 2 of [11] and figure 1 of [12]; the confinement time of tracer particles, for example, figure 3 of [13]; avalanches involving rearrangement rather than mass loss, for example, figure 2 of [8] and figure 2 of [10]; and avalanche size distributions [8–12, 15]. We note that, for obvious reasons of ease of detection, the existing experimental database on avalanches is weighted towards the subset of avalanches that result in particle loss from the edge of the sandpile. As remarked in [8], however, it can be argued that avalanches involving internal reorganization reflect a more direct response to the driving than do avalanches involving loss. In section 3 we shall turn to a class of confinement system, silos, where avalanches involving internal reorganization are the only kind permitted. It has also been pointed out [15] that energy dissipation in avalanches is a fundamental physical quantity which has not yet been subjected to the same level of experimental scrutiny as sand movement; we shall return to this in section 5.

Self-organized criticality has been observed [8, 13] in the distribution of avalanches in quasi-one-dimensional ricepiles, provided that the rice grains chosen are sufficiently elongated; there is qualitative agreement with a cellular automaton model. Measurements of the confinement and transport of tracer grains [13] in such piles have yielded important information on the underlying physics, some of which presumably carries over, perhaps in modified form, to the experimentally more widely known class of sandpiles with non-SOC avalanches. In particular, we note the following observations from [13]. Tracer grain transport combines convective and coherent components. The average transit time $\langle T \rangle$ of tracer grains scales as $L^{1.5\pm0.2}$, where L is the system size, so that the average velocity of tracer grains $\langle V \rangle \sim L/\langle T \rangle$ decreases with system size. Tracer grains thus possess information on system size, which implies the existence of long-range correlations within (and throughout) the ricepile. Transit time correlation studies indicate the existence of coherently moving regions whose size increases with L. While most tracer grains move close to the surface, some become buried much deeper and can only be released by infrequent major avalanches. Denoting by λ_L the depth of the active zone of the ricepile, where significant dynamics occurs, a statistically steady state implies $\langle V \rangle \lambda_L \sim$ constant. Hence $\langle V \rangle \sim 1/\lambda_L$, and it is pointed out that an increase in λ_L with L would be consistent with the experimental observation of decreasing average velocity. Furthermore, while the size of zones of coherent motion increases with L, observations are consistent with the hypothesis [13] that in large systems, large zones are broken up more often.

The observation that avalanches in real sandpile experiments, with exceptions [8, 13], tend not to replicate the SOC behaviour that emerges from idealized mathematical sandpile models poses several challenges. The most immediate is to construct a mathematical model which does replicate the observed behaviour of real sandpiles with non-SOC avalanches. One contribution is described in section 5 of this paper, where we attempt to model some consequences of the irregular properties of real sand (or rice), many of which are considered in [8–13]. These include the effect of finite grain size, grain geometry and grain inertia on sandpile packing and on avalanche dynamics, together with other more recondite properties of the granular state (a state which, like plasma, incorporates features of at least two conventional states of matter at a fundamental level; for reviews, see for example, [17, 18], and references therein). To this list we would add the distribution of stress within the sandpile, a subject which has been illuminated by recent experimental and theoretical advances in silo physics which are briefly reviewed in the next section.

3. Brief review of silo physics

By a silo, for present purposes, we mean a vertical cylinder containing randomly stacked granular matter. Such an object can be considered as a sandpile that is constrained not to have avalanches that transfer matter beyond its boundaries. Information about the distribution of force and matter within a silo thus bears directly on the structure of sandpiles. We also point out that if the lid is removed from a full silo, and the silo is then tilted away from the vertical, it becomes one of many possible realizations of the rotating drum sandpile. Several fascinating recent studies (see, for example, [19–21] and references therein) have illuminated the confinement physics of silos and some aspects of the link to sandpiles.

A clear summary of silo confinement phenomenology is provided by Claudin and Bouchaud in [20]. It has been known for a century that the weight supported by the base of a silo, W, is a small fraction of the weight of the matter that it contains, because much of the weight is taken up by friction against the side walls. The apparent weight W depends very sensitively on temperature, variations of a few degrees producing erratic and abrupt changes of plus-or-minus several per cent in W, while the relative volume change of the confined granular matter is only 10^{-5} to 10^{-4}. W is not a unique function of temperature, or of the mass confined—repeated silo fillings with identical numbers of beads produce variations in W exceeding 20%. Physically, the origin of these effects appears to lie in the fact [19, 20] that stress propagation within granular media is strongly inhomogeneous, resulting in narrow spatially extended stress paths that can transmit weight either to the walls of the silo, or to its base. Sensitivity of the network of stress paths to small perturbations may then account for the observed variability of W. This is modelled in [20] using a simple two-dimensional algorithm whereby each grain supports the weight of its two upstairs neighbours, and shares its own load randomly with its two downstairs neighbours. The random variable for the latter effect thus encompasses the (at a finer level of description, deterministic) microphysical effects of variation in, for example, grain size and shape. The model of [20] is completed by a threshold slip condition for loss of contact with one of the two near neighbours. This permits the formation of extended stress paths acting as arches, in a network whose configuration is highly sensitive to the precise value of the slip condition, see for example figure 3 of [20], and which changes as a consequence of 'static avalanches'. The silo aspect ratio (height/diameter) is also important, essentially because changes in stress paths occurring at a great vertical distance from the base are less likely to reach down to the base, so that weight will continue to be borne by the walls.

The model of [20] is restricted to two dimensions. Nevertheless, experiments by Liu *et al* [19] on a three-dimensional silo suggest that, as in two dimensions, the stress is concentrated along chains. Figure 1 of [19] shows a photograph of such stress paths, obtained by exploiting the stress-induced birefringence of glass beads packed in a silo and immersed in an index-matching fluid. An experimental demonstration of the arching consequences of the stress paths is shown in figure 2 of [19], obtained by placing carbon paper at the base of the silo, and exploiting the fact that the area of each mark on the carbon paper is proportional to the force exerted by the bead pressing upon it. The forces on the silo base in the region closest to the axis of cylindrical symmetry are weaker than those averaged across the entire base. While the distribution $P(w)$ of vertical forces w acting through the beads has the form $P(w) = C\exp(-Aw)$ in both cases, $C = 736$ and $A = 1.02N^{-1}$ in the central region, but $C = 988$ and $A = 0.64N^{-1}$ overall. Thus the combined effect of the stress paths is to shield the central region of the silo base from the full weight of the matter directly above it.

An apparently related experimental phenomenon is that the weight exerted by a conical sandpile on a surface has a minimum, not a maximum, below the apex. Wittmer *et al* [21] have recently extended the intuitive model of Edwards and co-workers [22, 23] involving arching due to lines of stress that are inclined to the vertical. The model of [21] incorporates a local constitutive relation between stress components which closes the basic system of equations and predicts macroscopic stresses that embody the physics of arching. There is good agreement (figure 3 of [21]) with experimental measurements of the weight distribution of conical piles of sand and of fertilizer, with no adjustable parameters.

Experimental and theoretical studies of silo confinement physics thus shed considerable light on the internal force structure of sandpiles, both conical and, as mentioned in the opening paragraph of this section, rotating drum. The full range of implications of internal force structure for avalanche dynamics has not yet been worked through. For example, it seems possible that rotating drum sandpiles may be affected by the geometry of the drum, unless the corresponding lengthscales and aspect ratio place the drum in some generalization of the 'large aspect ratio' limit considered in the silo context. The central importance of the interaction with the silo wall has already found a partial experimental analogue for rotating drums: in [10], the bottom of the tray was prepared by gluing down a variable number of layers of grains to give a rough surface. In the absence of this substrate, the granular bed slid frequently over the slick bottom surface of the tray. The importance of the initial stacking configuration is also particularly clear in silo physics. This suggests a possible source for differences between centrally fed conical, randomly sprinkled conical and rotating drum sandpiles which may create new (or through avalanches, wipe out initial) stacking information on different timescales.

The confinement properties of silos display salient features that are to some extent shared by tokamak plasmas, and may thus be candidates for identification as generic features of the physics of macroscopic confinement systems. These include: dependence on the geometrical parameters of the confinement system, such as the cylindrical aspect ratio; the importance of narrow extended self-organized structures within the medium; hysteresis; and irreproducability of macroscopic configuration parameters to an accuracy of better than 20% for equivalent initial settings. This question is distinct from other important topics, notably the wider applicability of avalanche models and experiments.

4. Some candidate aspects of tokamak phenomenology

It seems possible to extend the list of tokamak phenomena which may be susceptible to sandpile modelling beyond those proposed in the pioneering work of [3–7]. Carreras, Diamond, Newman and co-workers have already cited the following: profile resilience in low-confinement L-mode and ohmic discharges, linked to the principle of profile consistency [24–26]; evidence for the importance of disparate lengthscales (ion Larmor radius to minor radius) in L-mode confinement [27–29]; the possibility of universal indexes for measured broadband fluctuation spectra [30]; fast timescale energy propagation effects [31–33] which appear to be non-local and non-diffusive; and close relations between marginally stable and experimentally measured radial profiles [34, 35]. Before seeking to identify additional candidate phenomena for sandpile modelling, which is the main objective of this section, let us provide experimental examples of some of the types of phenomenon identified [24–35] by Carreras, Diamond, Newman and co-workers [3–6].

The phenomenon of profile resilience includes the striking effect that, in the discharges examined, the electron temperature profile tends towards an invariant shape irrespective of the radial distribution of heat input. Figure 5 of [36] shows two closely similar absolute values and profile shapes for the electron temperature, for discharges in the T-10 tokamak where the values of the power deposited within the plasma core ($r < 0.12$ m) differ by a factor of 3. This implies that the plasma self-organizes an effective core heat diffusion coefficient (heat flux/temperature gradient) differing by a factor of 4 (figure 6 of [36]) so as to maintain an almost invariant temperature profile in the two cases. In the DIII-D tokamak, no change in the electron temperature profile was observed [37] when the neutral beam power deposition profile was changed dramatically from centrally peaked to far off-axis (figure 3 of [37]). Also, the temperature profile was observed to maintain its strong centrally peaked form even when the heating was localized far from the plasma core: see, for example, figure 1 of [38] and figure 4 of [28]. This also implies non-diffusive transport including a component which moves heat up the electron temperature gradient. It seems reasonable to remark in this context that, at saturation, a conical sandpile of fixed radius will tend to maintain its height and shape irrespective of where, and (within limits) how rapidly, it is sprinkled with additional sand.

Experimental evidence for very rapid changes in transport properties extending over a broad radial region has been found for the transition from low to high confinement in tokamaks [39], and in the response of thermal diffusivity to applied heating in stellerators [40]. The measurements shown in figure 3 of [39] and figure 2 of [40] are particularly striking. Observations of inward propagation of cold heat pulses, created by sudden increases in impurity radiation at the plasma periphery, likewise imply sudden changes in transport properties over broad radial ranges in tokamaks; see, for example, figure 5 of [41] and figure 6 of [32]. We also highlight the recent analysis [42] of two striking experimental demonstrations that tokamak plasmas have very different confinement in very similar macroscopic states, with major changes occurring on very short timescales, see for example figures 4–6 of [42]; as we have noted, this is a generic feature of silo confinement [19, 20].

The cumulative impact of the experimental data considered thus far suggests, as Carreras, Diamond, Newman and co-workers [3–6] have done, that transport in a magnetically confined plasma can display elements of non-locality, rapid self-organization and criticality, with generic similarities to sandpiles and (we would add) silos. Let us now suggest additional candidate phenomena.

Perhaps the most promising phenomenon is that studied intensively by Berk, Breizman and co-workers [43–47]. The problem considered is the nonlinear evolution of kinetic instabilities exemplified by the classical bump-on-tail configuration and, most topically at present, by the interaction between fusion alpha-particles and discrete toroidal Alfvén eigenmodes (TAEs) supported by the background plasma. Theoretical studies predict a broad phenomenology for the modes, embracing pulsation, precursor oscillations, particle trapping, flattening of the distribution function, mode overlap, phase-space explosion, bursting and quasilinear diffusion. The regimes of occurrence of these various effects can be inferred theoretically, at least in outline, from the relative magnitude of a few key parameters of the model. A quantitative understanding of the statistics of these events is a central issue for magnetic fusion plasma physics. Specifically, TAE mode growth and induced expulsion of fusion alpha-particles may significantly reduce the collisional heating of the central plasma required for ignition, while increasing to undesirable levels the energy flux that loads the inner wall of the experiment.

Particularly catastrophic for alpha-particle confinement is the so-called domino effect [47]. It can occur when the unstable modes grow to be so closely spaced that they interact by phase-space overlap. It has been shown that the energy release caused by the overlap of closely spaced modes can produce overlap of more widely spaced modes, or the growth of modes which would otherwise be stable. This avalanche-type effect (see, in particular, figure 5 of [47]) can lead to corresponding rapid loss of alpha-particles from the system (figure 7 of [47]), and motivates the present suggestion that such events could be modelled using the self-organization of a sandpile algorithm whose properties are designed to replicate the physical system. Let us now consider how such a model might be established.

The basic kinetics of the Berk–Breizman system are contained (see, for example, equation (6) of [43]) in the one-dimensional Vlasov equation,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \sum_k \frac{eE_k}{m}\sin(kx - \omega_k t)\frac{\partial f}{\partial v} = S(v) - \nu(v)f \qquad (1)$$

describing the interaction between a minority population of energetic charged particles with distribution function $f(v, t)$ and a spectrum of waves with amplitudes E_k supported by the background plasma. The source and the sink on the right-hand side act to drive the system towards a zeroth-order (in E_k) distribution $f_0(v) = S(v)/\nu(v)$, the slope of which is assumed to be positive, $\partial f_0/\partial v > 0$, in the range where $v \simeq \omega_k/k$. The waves grow at the expense of the free energy contained therein. It follows from energy conservation that the wave energy density in each mode $W_k = \epsilon_0 E_k^2/4$ evolves according to

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + 2\gamma_d\right)W_k = eE_k\left\langle \int \sin(kx - \omega_k t)\, v\, f(x, v, t)\,\mathrm{d}v\right\rangle \qquad (2)$$

where $\langle\cdots\rangle$ denotes a spatial average and γ_d is the linear rate of damping caused by the background plasma. Given the zeroth-order distribution function $f_0(v)$, it follows from standard perturbation theory that the right-hand side can be written as $2\gamma_L W_k$, where

$$\gamma_L \propto \left.\frac{\partial f_0}{\partial v}\right|_{v=\omega_k/k}$$

is the linear growth rate; see, for example, equation (16) of [45].

Mode overlap plays an important role in determining phenomenology. Consider a single discrete mode driven unstable by resonant energy flow from an energetic particle population. It will grow according to equation (2) to reach a saturated amplitude and there will be a concomitant flattening of the velocity distribution function of the energetic particles in the neighbourhood of the phase velocity of the mode. In the simplest picture, two things could now happen. If the saturated mode does not overlap and interact with the neighbouring modes, it slowly damps away on the background plasma. On the other hand, if the flattened region extends to reach the flattened region of a neighbouring discrete mode, and this is repeated elsewhere, we enter the regime of quasilinear theory and substantial further flattening and energy release to the modes becomes possible by quasilinear diffusion. Where modes overlap, the averages of equations (1) and (2) simplify to

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}D\frac{\partial f}{\partial v} + S(v) - \nu(v)f$$

$$\frac{\mathrm{d}W_k}{\mathrm{d}t} = 2(\gamma_k - \gamma_d)W_k$$

with D the quasilinear diffusion coefficient and $\gamma_k \propto \partial f/\partial v$ taken at $v = \omega_k/k$. These equations, which are taken from equations (19) and (22) of [45], are formally identical to equations (1*a*) and (1*b*) of [48], where they were proposed as a generic model for a driven dissipative system with self-organized criticality. The only essential difference is the way the growth rate γ_k is constructed.

Returning to the original system, equations (1) and (2), analytical theory predicts, and numerical simulation confirms [43–47], that in the limit $\nu \ll \gamma_d \ll \gamma_L$ the system is violently unstable and behaves as follows. The distribution function evolves slowly towards f_0 on the longest time scale $1/\nu$. Once its slope becomes large enough at some velocity which is nearly resonant with one of the waves, $v \simeq \omega_k/k$, the latter grows rapidly. This may occur long before the asymptotic state f_0 is reached. If the velocity-space separation from the nearest neighbouring mode is small enough, the two waves start overlapping, and the distribution function is completely flattened in this region. The modes may then grow further and overlap with the next neighbour, and so on, leading to an avalanche. Large-scale flattening then occurs as a consequence of quasilinear diffusion. It is important here that the instability grows and results in mode overlap and flattening of the distribution function, on the fastest time scale which is much shorter than was required to build up the slope in the first place. On an intermediate timescale, the mode energy damps on the background plasma while, on the longer timescale determined by the source $S(v)$, the forces (source and sink terms) responsible for creating free energy in the energetic particle distribution slowly drive the system back towards criticality.

This separation of time scales makes it possible to translate the avalanche dynamics to sandpile-type rules. The kth position in the sandpile corresponds to the mode at $v_k = \omega_k/k$, and the height of the pile to the distribution function $f(v_k)$. The modes are, for simplicity, assumed to be equidistant. If the local slope at a given point is large enough, and exceeds the critical gradient, sand is redistributed in such a way as to produce local flattening. The redistribution is instantaneous because of the fast growing nature of the instability, and thus matches a generic feature of mathematical sandpile algorithms. The way that sand is added is determined by the longest-timescale physics. This suggests gradual addition at all positions simultaneously, in such a way as to steadily increase the positive slope of the distribution at all positions. We are thus led to the following generalized sandpile rules for the system:

(i) Add sand gradually and continuously at *all* positions, increasing the slope by uniform amounts at all positions until it becomes critical somewhere.

(ii) Redistribute sand at that location, flattening the pile completely there.

Rule (i) corresponds to the particularly simple limit $\partial f/\partial v \ll \partial f_0/\partial v$, where the slope grows at a constant rate at all positions. It also represents a mathematical idealization of the behaviour of a sandpile that is gradually tilted, as in the experiments with sand in a rotating drum [9, 10], with the slope normalized to the angle of repose which the pile assumes after each avalanche. In the present paper, we do not propose to go further and analyse a particular sandpile model for the Berk–Breizman system. Our essential point is that it seems possible to construct a *prima facie* case for a link between sandpiles (modelled or real) and some aspects of Berk–Breizman dynamics.

A second additional candidate is provided by reconnection. As already mentioned, Lu and Hamilton [7] have investigated the hypothesis that solar flares arise from an avalanche of magnetic reconnection events, by comparing observed statistics for the occurrence of solar flares of different magnitude with the avalanche statistics arising from a sandpile-type model that is intended to incorporate the key features of reconnection physics. In the model, a sequence of random additions $\delta \boldsymbol{B}$ is made to the magnetic field $\boldsymbol{B}$ at random positions on a three-dimensional grid, simulating the effect of random twisting of the magnetic field by photospheric circulation. When the local magnetic field gradient exceeds a critical value, 'reconnection' occurs, involving vectorial cancellation of the local field and its redistribution to neighbouring grid points; this may drive the magnetic field at a neighbouring point unstable against reconnection, leading to an avalanche effect. The statistical distribution of magnetic energy releases obtained from avalanches in this model exhibits self-organized criticality scaling, with a power law that matches that of the solar flare data. This is encouraging for the sandpile approach to this type of phenomenon, but one may have reservations about the immediate applicability of the model of [7] to fusion plasmas. The choice of magnetic field gradient for the criticality condition does not easily carry over into the tokamak scenario, where reconnection is driven by gradients in the current profile and occurs preferentially at specific spatial locations (rational surfaces). Also, the magnetic field in the sandpile model of [7] does not satisfy $\nabla \cdot \boldsymbol{B} = 0$. Although this may not affect the avalanche dynamics, it is unclear that the square of the vector field in the model represents magnetic energy, and this may have consequences for the power-law scaling observed.

The approach to reconnection and disruptions in tokamaks pioneered by Carreras, Hicks and co-workers [49, 50] also appears to have potential for sandpile modelling. Here, nonlinear coupling of growing tearing modes of different helicity leads to global destabilization. Magnetic islands associated with the modes grow and, if they overlap, destabilize each other further. This results in a rapidly increasing volume of stochastisized magnetic field, flattening the temperature profile and changing the current profile that provided the free energy for the initial instability. The mode overlap displays that emerge from detailed numerical calculations of this process (for example, figure 6 of [49] and figure 14 of [50]) share features with those obtained for the Berk–Breizman system, for example, figure 5 of [47] mentioned above. We may note, in particular, the sequential overlap of the neighbouring $(5, 3), (2, 1), (5, 2)$ and $(3, 1)$ modes in figure 14 of [50], which comprises an apparent domino effect. In a sandpile model of this process, each node would correspond to a rational surface of the equilibrium magnetic field; the height of the pile would correspond to the current density; the critical gradient would reflect the value of $\mathrm{d}j/\mathrm{d}r$ that triggers tearing; and the redistribution algorithm would reflect local flattening of the current gradient due to magnetic stochastisization consequent on tearing and magnetic island growth.

It might be possible to construct a sandpile model in terms of discretized current filaments. The statistics of current filaments and their links to tokamak current profiles and transport have been studied by Taylor [51]. Kadomtsev [52], in developing a theory in which the plasma evolves towards a state of contiguous nested chains of magnetic islands, has pointed out that a description in terms of current filaments is also possible.

Let us now turn to some considerations arising from [8, 13]. First, there is the striking qualitative dependence (avalanches displaying SOC or not) of the global transport on what we may term the rice ‘isotope’ (actually *naturris*, *grøtris* or *middagsris*). We refer to [53, 54] for experimental reviews of the isotope effect on tokamak confinement. Simulations suggest [55] that this may arise because of a differential effect on the spectral structure of the self-generated turbulence. Observations [8, 13] indicate that the ricepile ‘isotope’ effect arises from the differential consequences of grain aspect ratio for the self-organized convective and coherent transport. The use of tracer grains [13] to illuminate the physical processes and structures involved in ricepile transport parallels the recent use of test particles in simulations of transport in strongly turbulent plasmas with highly evolved coherent nonlinear structures [56, 57]. Further parallels exist in the questions concerning the scaling with system size of properties of extended coherent structures in ricepiles [13]. As mentioned in section 2, the scaling of coherent structure size and resilience against disruption emerge as central issues from ricepile experiments. Closely analogous questions are important for transport in tokamaks, see for example, [58].

There may also be a link between edge-localized modes (ELMs) [59, 60] in tokamaks and the self-organized behaviour observed in sandpiles. ELMs are commonly thought to be triggered by steep edge gradients. The latter are, in turn, modified by the ELMs ejecting particles and heat into the scrape-off layer, thus probably leading to a self-organized state. The concomitant quasiperiodic bursting hydrogen radiation signal bears a resemblance to the avalanches observed in [9] in a slowly rotated drum filled with sand. Type I ELMs in DIII-D H-modes have been observed [59] to regulate the plasma density at a value determined by a stability limit. They impose a density limit in the presence of beam fuelling (see, for example, figure 1 of [59]), and ELM frequency rises in proportion to beam power. Furthermore, attempts to raise the plasma density by gas puffing are nullified by increased ELM frequency, which scales linearly with the fuelling rate. These phenomena have a clear sandpile analogy: if one raises the rate of sand addition, the rate of avalanching will increase in proportion. Thus, the ELMs would be seen as avalanches that maintain the density profile and fuelling in a state of self-organized balance. The statistics of ELMs (especially amplitude and separation in time) could repay further study in this context. Following the discussion in section 2, we note that ELM statistics need not necessarily exhibit SOC for a sandpile-type algorithm to be potentially valid.

Finally, we remark that some of the motivation for modelling sandpiles and related systems lies in the hope that predictive principles (in particular, for earthquakes and their precursor tremors; see, for example, [12, 61, 62]) may emerge. There would be clear benefits if such principles were transferable to tokamaks, and available to help pre-empt undesirable phenomena in real time.

5. A model for experimental sandpiles

As discussed in section 2, experimental sandpiles do not usually behave like the theoretical paradigm of SOC introduced by Bak *et al* [1]. This does not, however, rule out other possibilities for modelling sandpiles by cellular automata. A broader spectrum of sandpile-type models may also be desirable for future applications, given the increasing range of physical phenomena for which the sandpile approach appears potentially relevant. In a recent paper we have developed one such model [63], which we now briefly describe.

It seems likely that any model capable of replicating the observed behaviour of a real sandpile will need to incorporate smearing out of the ideally sharp critical gradient condition, arising from the irregular properties of real sand (or rice). In the present model, this is achieved by adopting a probabilistic view of the sandpile. The pile itself consists of a discrete number of positions, labelled by $n = 1, \ldots, N$, each with a certain slope, $z_n = h_n - h_{n+1}$, which is normalized to zero at the angle of repose. Any change in slope $\mathrm{d}z$ (either by external interference or by internal redistribution) is associated with a probability $\mathrm{d}p$, depending on the local slope z, that the pile becomes locally unstable,

$$\frac{\partial p}{\partial z} = f(z).$$

Since instability becomes increasingly likely as the pile becomes steeper, $f(z)$ should be a monotonically increasing function, vanishing at the angle of repose $z = 0$. A simple model is

$$f(z) = z^y \tag{3}$$

with $y > 0$ an adjustable parameter defining the sharpness of the critical gradient.

Once instability occurs somewhere in the pile, the slope relaxes locally to the angle of repose by redistributing sand to the next position. This process is regarded as instantaneous, and if several adjacent positions are simultaneously unstable the pile is flattened over the entire unstable region. Adding sand can be done in different ways, for example by

(i) Central fuelling: adding single grains of sand at the top of the pile, i.e. $z_1 \to z_1 + g$ with g the grain size.

(ii) Sprinkling: adding single grains of sand at random positions, uniformly distributed over the pile, i.e. $z_n \to z_n + g$, $z_{n-1} \to z_{n-1} - g$, with n random and uniformly distributed over $1 \leqslant n \leqslant N$.

(iii) Tilting: increasing the slope continuously by equal amounts everywhere, i.e. $z_n \to z_n + t$ for all n, where t denotes dimensionless time.

In each case the slope is increased by one of these external means until some position becomes unstable. The redistribution rules are then applied until the pile has relaxed to a new stable state, after which the procedure is repeated.

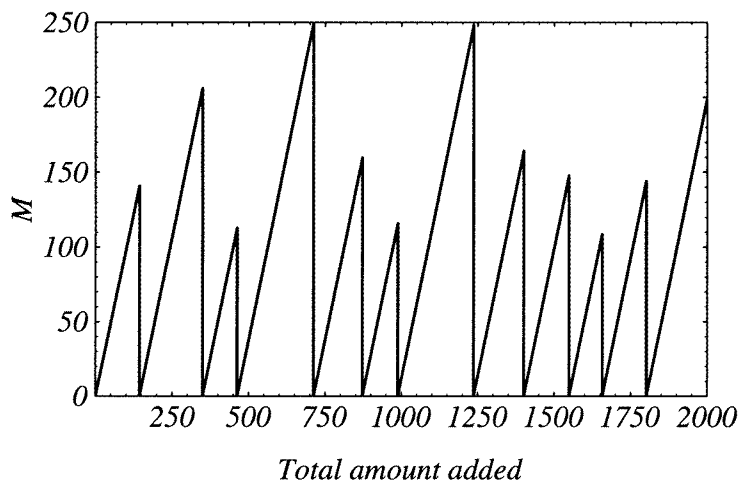

Figure 1. The total amount of sand M (above the angle of repose) in the pile versus the total amount of sand added to the pile by central fuelling of particles. The length of the pile is $N = 50$, the exponent in equation (3) is $y = 1$ and the size of the sand grains is $g = 0.01$.

Largely independently of which type of fuelling is used, the total amount of sand

$$M = \sum_{n=1}^{N} n z_n = \sum_{n=1}^{N} h_n$$

typically varies with the amount of added sand as shown in figure 1, where M increases linearly until a major avalanche occurs. The pile then relaxes to the angle of repose, and all excess material leaves the system. Small avalanches, resulting in some, but not all, sand above the angle of repose leaving the pile are infrequent. All avalanches are of the same order of magnitude, i.e. no power law or other broad distribution of avalanche sizes is observed. This is in agreement with most experiments on real sandpiles, in particular with [9, 12] and the largest of the sandpiles in [11] (see figure 1 in [12] and figure 2(d) in [11]). Changing the parameter y does not influence the results much, regardless of the way the pile is fuelled. If $y \gg 1$, instability is very unlikely unless $z \geqslant 1$. For large y, there is therefore a lower limit to the avalanche sizes.

While SOC is not observed in the material that leaves the sandpile, a rather different picture emerges if one studies the *internal* dynamics of a pile that builds by central fuelling,

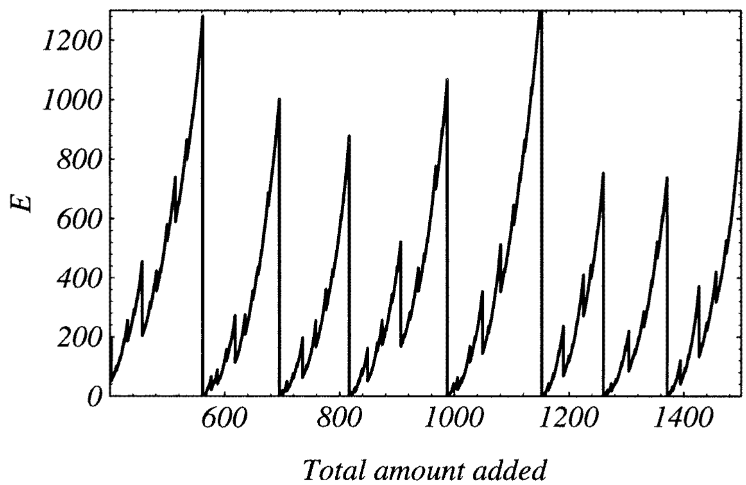

Figure 2. Potential energy E stored in the sandpile versus the total amount of sand added by fuelling at the centre; $y = 1$, $N = 50$, $g = 0.001$.

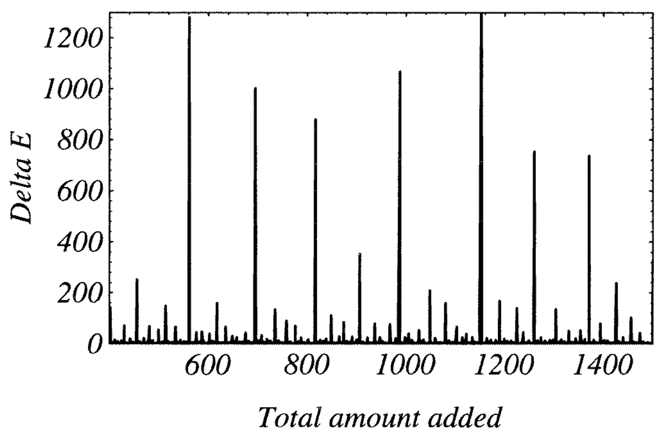

Figure 3. Potential energy released in the avalanches in figure 2.

and this *does* involve SOC. Figure 2 shows the evolution of the potential energy

$$E = \sum_{n=1}^{N} h_n^2$$

in the sandpile when fuelled with small grains at the centre. The energy increases quadratically with the amount of sand added, but is frequently interrupted by minor avalanches. What happens is that when the pile is empty and the fuelling begins, a small pile is first established near the centre since this is the only place where sand is added. The pile then grows and spreads by a series of avalanches propagating outwards. These events occur on all scales until the front of the pile reaches the edge, and a major avalanche emptying the entire pile soon occurs. The time series of avalanches, shown in figure 3, is strongly reminiscent of figure 2(c) in [8], which appears to be the only report of an experiment where energy dissipation has been recorded. Figure 4 shows a logarithmic plot of the avalanche energy distribution. Most of the distribution is well approximated by a power law, shown as a full curve, which is the signature of SOC. The behaviour is insensitive to the size N of the sandpile and the grain size g. In other words, the internal dynamics of the growth of the sandpile is self-organized and critical, while this manifestation of SOC is invisible to an observer watching only what leaves the system.

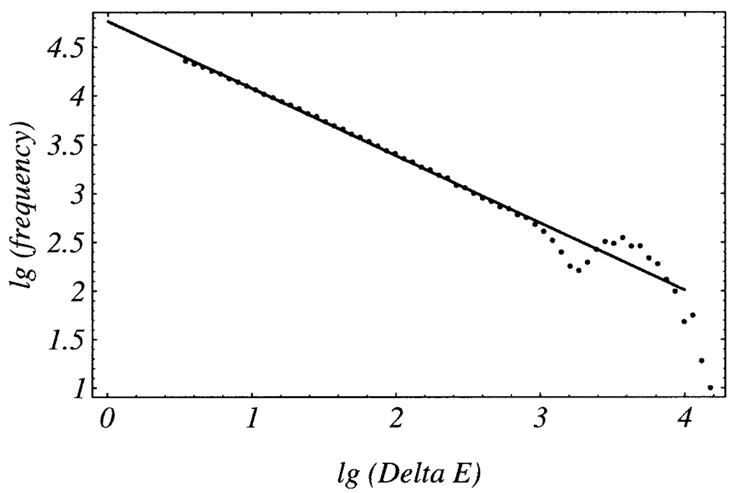

Figure 4. Distribution of avalanche energies (dots) in a centrally fuelled pile; $y = 1, N = 100, g = 0.001$.

6. Conclusions

The material presented in this paper tends, in our view, to support the suggestion that the phenomenology of tokamaks, sandpiles and related systems may display common features that are generic to macroscopic confinement physics. We have explained how several key aspects of tokamak phenomenology (most notably, perhaps, ELMs and Berk–Breizman dynamics), additional to those already identified by Diamond, Carreras, Newman and co-workers, may lie within this zone of commonality. This implies that advances in modelling sandpiles (especially real, experimentally observed ones) may also benefit the wider field. However, it should be noted that the behaviour of sandpiles, silos and theoretical models thereof is sometimes sensitive to the experimental set-up or the details in the modelling. Clearly, there remain significant questions to resolve. Perhaps the most important concerns the strength, or otherwise, of the link between sandpiles and tokamak physics since it is difficult to derive sandpile rules rigorously from the underlying plasma physics. This issue can be illustrated by a thought experiment, as follows. Suppose that one were to discover a mathematical sandpile algorithm (fuelling, criticality and redistribution) that, in a sharply defined parameter regime, gave greatly enhanced confinement with steep edge gradients and a flat central profile, reminiscent of a tokamak H-mode. Under what circumstances could this 'sandpile H-mode' be regarded as a genuine and interesting emergent phenomenon, as distinct from a 'rigged' effect clearly implicit in the sandpile algorithm? The answer must depend on the strength of the link between the model and basic plasma physics, the level of simplicity of the algorithm, and the robustness of the phenomenon. A related point is that the role of SOC is not yet entirely clear either in actual confinement systems or in their modelling. Here, the results of the sandpile model presented in section 5 may be important, by showing that SOC can play a hidden role in a sandpile-like system, governing its internal dynamics while being invisible in the output signal from the system.

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