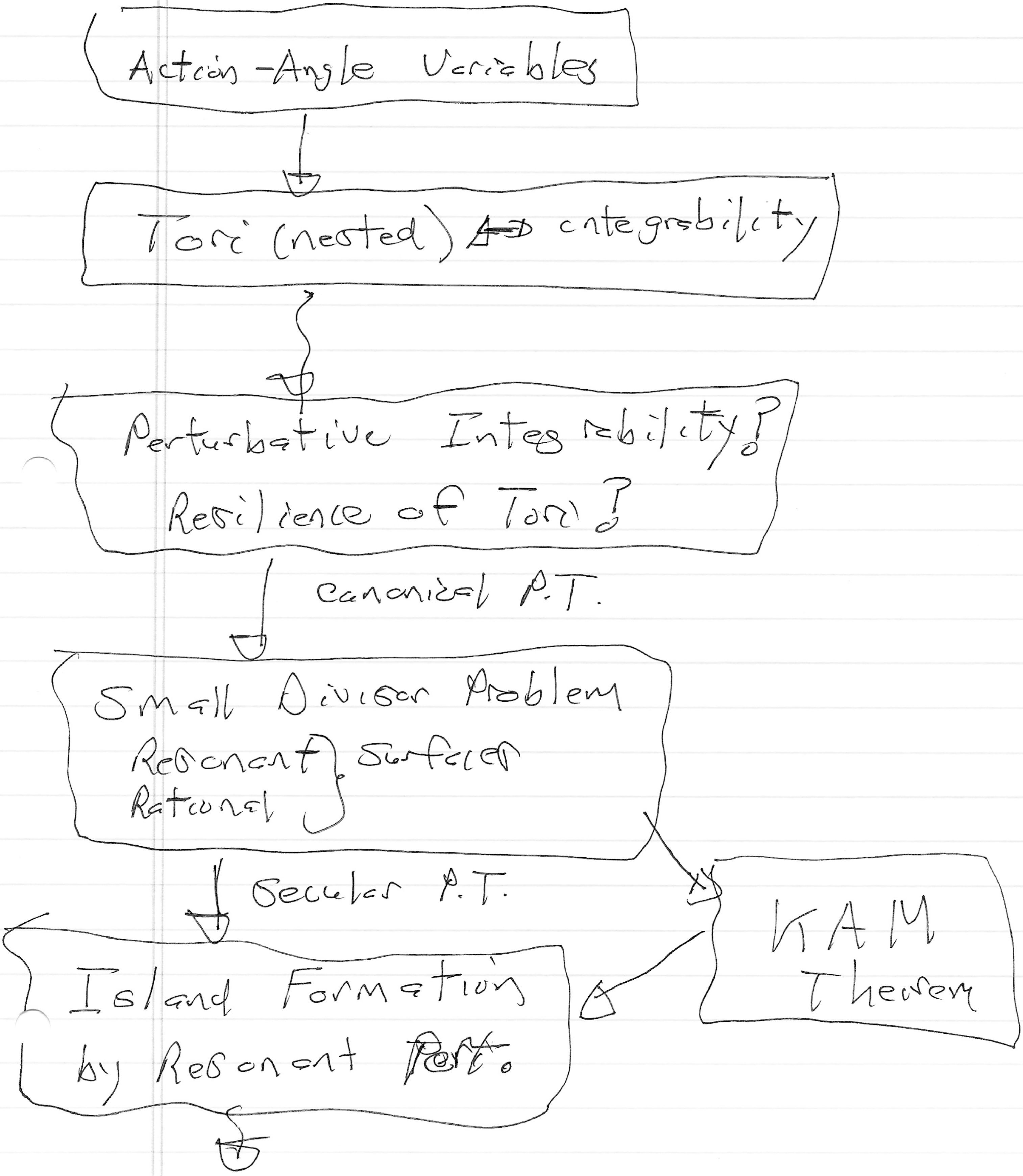
→ Flow Chart


Action-Angle Variables
Tori (nested) ⟺ integrability
Perturbative Integrability?
Resilience of Tori?
canonical P.T.
Small Divisor Problem
Resonant
Rational
Surfaces
Secular P.T.
KAM Theorem
Island Formation
by Resonant Pert.

conceptual step

$\boxed{\text{Chirikov Crit. / Overlap}}$

study of systems ~~↔~~ standard Map

$\boxed{\text{Chaos, Stochasticity}}$

Calculation in Chaotic Regime ? ?

Calculating in Chaotic Regime I

→ A First Introduction to Stat. Mech.

→ have discussed:

- problem of perturbative integrability
- 'small divisors' and island formation
- KAM Theorem
- Chirikov criterion and onset chaos

now begs question: $\boxed{\text{How calculate in chaotic regime ??}}$

⇒ Statistical Mechanics !!! ⇒ { seek calculate pdf of orbits

- chaos ⟺ deterministic, random process

- approach via methodology of random processes:
  - Mean Field / Quasilinear Theory
  - Fokker-Planck Theory

- Key approximation:
  - fundamentally, orbit is stochastic
  - but treat perturbation as random, even though it is <u>not</u>.

i.e.
- off example, re: $\theta \rightarrow$ phase
- 'know' $\langle \tilde{B}_r^2 \rangle_{m,n}$ but not $\tilde{B}_{r_{m,n}}$

i.e. $\langle \tilde{B}_{r\,m,n}^{2} \rangle$ fixed

$\tilde{B}_{r\,m,n} = 0$

$\Rightarrow$ $\tilde{B}_r$ as Gaussian random variable (via phase)

$\Rightarrow$ RPA

Random Phase Approx

– Seek Calculate:

(a) – diffusion coefficient $D_J \to \langle (\Delta J)^2 \rangle / \Delta t$

$\rightsquigarrow$ basic measure chaos

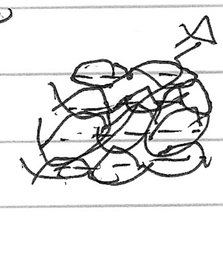
(b) – orbit divergence rate $\Rightarrow$ i.e. Lyapunov exponent h.

(a) Field Line Diffusion

– Hierarchy Truncation Mean Field Theory

How far how fast do orbits 'hop' in radius?

$f \equiv$ pdf of orbit in r, θ, z



i.e. recall: $\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{dz}{B_z}$

strategy: $\frac{\partial}{\partial z}\langle f \rangle$

obviously f constant along field,

so:

$$B \cdot \nabla f = 0$$

total f conserved along field

$\Leftrightarrow$ Liouville Eqn.

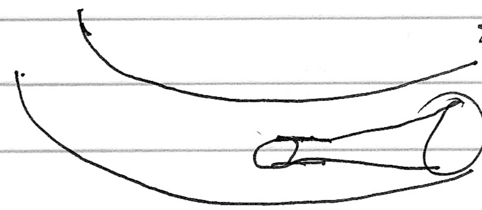
$$\Rightarrow \quad \frac{\partial}{\partial z} f + \frac{B_\theta}{B_z}\frac{1}{r}\frac{\partial f}{\partial \theta} + \frac{B_r}{B_z}\frac{\partial f}{\partial r} = 0$$

$$\begin{cases} B_r = \tilde{B}_r \\ B_\theta = \underbrace{\langle B_\theta(r) \rangle}_{\text{toroidal current}} + \underbrace{\tilde{B}_\theta}_{\text{pert.}} \end{cases} \qquad \nabla_\perp \cdot \tilde{B}_\perp = 0$$

and

$$f = \underbrace{\delta f}_{\substack{\text{fluctuation} \\ \text{induced by } \tilde{B}}} + \underbrace{\langle f \rangle}_{\substack{\text{mean dist.} \\ \Updownarrow \\ \text{coarse graining!}}}$$

$$\begin{cases} \delta f = \delta f(\underbrace{\phi}_{\text{fast}}, \theta, r), \quad \phi \leftrightarrow z \\ \langle f \rangle = \langle f(\underbrace{z}_{\text{slow}}, r) \rangle \end{cases}$$



$$\Rightarrow$$

$$\frac{\partial f}{\partial z} + \frac{\langle B_\theta \rangle}{B_z r}\frac{\partial}{\partial \theta} f + \frac{\partial}{\partial r}\left(\frac{\tilde{B}_r f}{B_z}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\tilde{B}_\theta f}{B_z}\right) = 0$$

Now, – $\langle \ \rangle = \langle \ \rangle_{\theta,\phi}$

but retain slow dependence

– alternatively, ensemble.

→ averaging:

$$\frac{\partial \langle f\rangle}{\partial z} + \cancel{\left\langle \frac{\langle B_\theta\rangle}{B_z r} \frac{\partial}{\partial \theta} f \right\rangle} + \frac{\partial}{\partial r}\left\langle \frac{\tilde{B}_r}{B_z} \delta f \right\rangle = 0$$

→ Mean Field Equation:

$$\boxed{\frac{\partial}{\partial z}\langle f\rangle + \frac{\partial}{\partial r}\left\langle \frac{\tilde{B}_r}{B_z} \delta f \right\rangle = 0}$$

{ tracks slow variation in z due radial flux $\langle f\rangle$

b.- $$\frac{\partial}{\partial z}\langle f\rangle + \frac{\partial}{\partial r}\Gamma = 0$$

div. flux → continuity equation

- hierarchy pblm.

i.e. $\delta f \Leftrightarrow \tilde{B}_r$

$$\frac{\partial}{\partial z}\delta f + \frac{\langle B_\theta\rangle}{B_T}\frac{\partial}{\partial y}\delta f = -\frac{\tilde{B}_r}{B_T}\frac{\partial \langle f\rangle}{\partial r}$$

$$\tilde{B}_r = \sum_{m,n} \tilde{B}_{r\,m\,n} e^{i(m\theta - n\phi)}$$

$$= \sum_{k_z, k_\theta} \tilde{B}_{\underline{k}} e^{i(k_\theta y - k_z z)}$$

(slab)

so $\frac{\partial}{\partial z} \langle f \rangle \sim \mathcal{O}(\delta f^2)$

$\mathcal{O}(\tilde{B}_r^{\,2})$

$\frac{\partial}{\partial z} \langle \delta f^2 \rangle \sim \mathcal{O}(\delta f^3)$

$\Downarrow$

$\left\{ \begin{array}{l} \infty \text{ hierarchy} \\ \text{of moments} \end{array} \right.$ i.e. eqn. bilinear

how truncate?

$\Downarrow$

- mean field theory / quasilinear theory!
- compute $\left\langle \frac{\tilde{B}_r}{B_0} \delta f \right\rangle$ break average!

  by substituting linear response δf into flux – why??
- yields a mean $\Gamma(r)$ $\rightarrow$ radial flux due stochastic wandering of lines.
- closes hierarchy, as:

$$\Gamma = -D_M \frac{\partial \langle f \rangle}{\partial r}$$

Fickian flux. $D_M \Leftrightarrow$ counterpart of $D \simeq K^2/4$ for st. map

$\Rightarrow$ diffusion equation! for $\langle f \rangle$

$$\Gamma = \sum_{m,n} \frac{\tilde{B}_{r_{-m,-n}}}{B_0} \delta f_{m,n}$$

$$= \sum_{\underline{n}} \frac{\tilde{B}_{r_{-\underline{n}}}}{B_0} \delta f_{\underline{n}}$$

Now,

$$i\left(k_y \frac{\langle B_\theta \rangle}{B_T} - k_z\right) \delta f_{\underline{n}} = -\frac{\tilde{B}_{r\underline{n}}}{B_0} \frac{\partial \langle f \rangle}{\partial r}$$

$\Rightarrow$

$$\delta f_{\underline{n}} = \frac{i}{\left[k_z - k_y \frac{\langle B_\theta \rangle}{B_T} + i\epsilon\right]} \left(-\frac{\tilde{B}_{r\underline{n}}}{B_0} \frac{\partial \langle f \rangle}{\partial r}\right)$$

causality (damping at $z \to -\infty$)

$\Rightarrow$

$$\frac{i}{x + i\epsilon} = i\left(\frac{P}{x} - i\pi \delta(x)\right)$$

$$= \cancel{\frac{iP}{x}} + \pi \delta(x)$$

$\Rightarrow$

$$\Gamma = -\sum_{\underline{n}} \left|\frac{\tilde{B}_{r\underline{n}}}{B_0}\right|^2 \pi \delta\left(k_z - k_y \frac{\langle B_\theta \rangle}{B_T}\right) \frac{\partial \langle f \rangle}{\partial r}$$

ω

$$\boxed{\begin{aligned} \Gamma &= -D_M \frac{\partial \langle f \rangle}{\partial r} \\ D_M &= \sum_k \left| \frac{\tilde{B}_{r_k}}{B_0} \right|^2 \pi \delta\left(k_z - k_y \frac{\langle B_\theta \rangle}{B_T}\right) \\ &= \sum_{m,n} \frac{|\tilde{B}_{r_{m,n}}|^2}{B_0^2} R \pi \delta\left(n - \frac{m}{q(r)}\right) \end{aligned}}$$

$D_M \equiv$ field line diffusion coefficient

and Q-L / Mean Field Eqn.:

go to 6

$$\boxed{\frac{\partial \langle f \rangle}{\partial z} = \frac{\partial}{\partial r} D_M \frac{\partial \langle f \rangle}{\partial r}}$$

$\langle \delta r^2 \rangle \sim D_M z$

N.B. $D_M \sim L$.

- For standard MHD:

$D \simeq K^2$, $\ell_{ac} \simeq 1$

parallel coherence length of scattering field.

- here $D \sim \left| \frac{\delta B_r}{B} \right|^2$, $\ell_{ac} \sim \frac{1}{\Delta |k_\parallel|}$

i.e. $\frac{\partial \langle f \rangle}{\partial Z} = \frac{\partial}{\partial r} D_M \frac{\partial}{\partial r} \langle f \rangle \Rightarrow \langle \delta r^2 \rangle \sim D_M Z$

- (very impt.)
  → origin of irreversibility →
  Chaos!, due island overlap.

→ why diffusion?

- trajectory can 'hop' either direction radially → net flux set by $\partial \langle f \rangle / \partial r$.

- diffusion measures:
  mean square displ.
  $\langle (\Delta r)^2 \rangle / \Delta Z$ $\left(\rightarrow \frac{\langle (\Delta x)^2 \rangle}{\Delta t} \right)$

random walk ⇕ chaos!

Δr → step size, radial
ΔZ → Z increment (time step).

- why linear response valid?
  $l_{ac} < l_\perp$ (tbd) $Ku < 1$

i.e. scattering field pattern decorrelates rapidly ⇒ ensures:

$$\frac{l_{ac}}{\Delta_r}\frac{\tilde{B}_r}{B_0} < 1 \quad \Rightarrow \text{Kubo \#} \quad K_u < 1.$$

- What does Kubo # mean?

$\Delta_r \rightarrow$ radial scale of scattering field

i.e. $\tilde{B}_{r_{m,n}}(r) \Leftrightarrow$ radial scale

i.e. $\tilde{B} = \tilde{B}(r/\Delta)$

$\delta r \rightarrow$ scattering excursion in radius

i.e. $\frac{dr}{dz} = \frac{\tilde{B}_r}{B_0} \Rightarrow \delta r \sim \int dz \frac{\tilde{B}_r}{B_0}$

$l_{ac} \rightarrow$ parallel coherence length of scattering field $\quad \sim l_{ac}\frac{\tilde{B}_r}{B_0}$

then $\frac{\delta r}{\Delta} < 1$ → weak scattering ($K < 1$)

→ many kicks in 1 radial scale of scatterer.

→ linear response o.k.

$\frac{\delta r}{\Delta} > 1$ → strong scattering ($K > 1$)

→ linear response fails.

n.b. : $\Rightarrow \underline{k} \cdot B_0 = k_y \langle B_0 \rangle + k_z B_T$

$$\rightarrow \quad = k_y \frac{\langle B_0 \rangle}{B_T} - k_z .$$

$\rightarrow l_{ac} \sim \dfrac{1}{\Delta |k_\parallel|}$ $\rightsquigarrow$ inverse bandwidth of parallel wave-number spectrum.

$\Downarrow$ {width of $k_\parallel$ spectrum

so, back to why linear response?

$\rightarrow$ small kick, relative to radial scatterer correlation length!

$\rightarrow$ kick $\delta r \sim l_{ac} \dfrac{\tilde{B}_r}{B_0}$, $\dfrac{dr}{dz} = \dfrac{\tilde{B}_r}{B_0}$

$\delta r < \Delta_\perp \Rightarrow K_u < 1$ again!

$\rightarrow D \sim \langle \delta r^2 \rangle / \Delta z$

$\delta r \sim l_{ac} \dfrac{\tilde{B}_r}{B_0}$

$\Delta z \sim l_{ac}$

$\Rightarrow D \sim l_{ac} \langle (\tilde{B}_r / B_0)^2 \rangle \rightarrow D_M$.

check:

$$k_r^2 D_r \quad \text{vs} \quad \Delta |k_\parallel|$$

$\Rightarrow$

$$\frac{D_r}{\Delta_r^2} < |\Delta k_\parallel|$$

$\Rightarrow$

$$\frac{(\delta r/\delta z)^2}{\Delta_r^2} l_{ac} < \Delta |k_\parallel|$$

$$\left(\frac{\tilde{B}_r}{B_T^2} \frac{l_{ac}}{\Delta_r} \right)^2 < 1 \quad \Rightarrow \quad \text{Kubo}$$

- General structure of standard map paradigm

$$\frac{\partial f}{\partial t} + \underline{\omega}(\underline{J})\cdot\nabla_{\underline{\theta}} f - \frac{\partial \tilde{H}(J,\theta)}{\partial \underline{\theta}}\cdot\frac{\partial f}{\partial \underline{J}} = 0$$

$$\frac{\partial f}{\partial z} + \frac{\langle B_\theta\rangle}{B_T\, r}\frac{\partial f}{\partial \theta} + \frac{\tilde{\underline{B}}}{B_T}\cdot\underline{\nabla} f = 0$$

$$\langle \omega(J)\rangle \iff \frac{1}{R q(r)}$$

$$J \iff r$$

$$m\tilde{H} \iff \frac{\tilde{B}}{B} \qquad \tilde{H} \iff \tilde{A}$$

and

$$\frac{\partial \langle f\rangle}{\partial t} = \frac{\partial}{\partial \underline{J}}\cdot\left\langle \frac{\partial \tilde{H}(\underline{J},\theta)}{\partial \underline{\theta}}\,\delta f\right\rangle$$

and $$-i(\underbrace{\omega}_{\Omega} - \underbrace{\underline{m}\cdot\underline{\omega}(J)}_{\text{Key resonance.}})\,\delta f_{\underline{m}} = \left(\frac{\partial \tilde{H}}{\partial \underline{\theta}}\right)_{m}\frac{\partial\langle f\rangle}{\partial \underline{J}}$$

$$-i\left(\Omega - \underline{m}\cdot\omega(\underline{J})\right)\delta f_{\underline{m}} = i\underline{m}\,\tilde{H}_{\underline{m}}\,\frac{\partial\langle f\rangle}{\partial \underline{J}}$$

need overlap of $\frac{\Omega}{m} = \omega(J)$ resonances

$$\delta f_m = \frac{i}{\Omega - m\omega(J)}\, i m \tilde{H}_m \frac{\partial\langle f\rangle}{\partial J}$$

$$\cong \pi\delta\left(\Omega - m\omega(J)\right) i m \tilde{H}_m \frac{\partial\langle f\rangle}{\partial J}$$

so

$$\boxed{\begin{aligned}\frac{\partial\langle f\rangle}{\partial t} &= \frac{\partial}{\partial J} D_J \frac{\partial\langle f\rangle}{\partial J}\\ D_J &= \sum_m m^2|\tilde{H}_m|^2\,\pi\delta\left(\Omega - m\omega(J)\right)\end{aligned}}$$

- field line correspondence with this general structure is obvious.

Now, another length scale emerges, from considering decorrelation of trajectory from linear one, due scattering in J

i.e.

$$\frac{d\theta}{dt} = \omega(J)$$

$$\frac{d}{dt} \underset{\text{excursion}}{\delta\theta} = \frac{\partial \omega}{\partial J} \delta J$$

$\Rightarrow$

$$\delta\theta \sim \int^t \frac{\partial \omega}{\partial J} \delta J$$

$$\langle \delta\theta^2 \rangle \sim \left(\frac{\partial \omega}{\partial J}\right)^2 \int^t \int^t \langle \delta J^2 \rangle$$

but $\langle \delta J^2 \rangle \sim D_J t$

$$\boxed{\langle \delta\theta^2 \rangle \sim \left(\frac{\partial \omega}{\partial J}\right)^2 D_J t^3}$$

$\rightarrow$ enhanced decorrelation from scattering of action

i.e. not $\sim t$

For F-O-M of scattering: use m of scatterer field.

$$m^2 \langle \delta\theta^2 \rangle \sim 1 \sim m^2 \left(\frac{\partial\omega}{\partial J}\right)^2 D_J t^3$$

$$\Rightarrow \; 1/\tau_c \sim \left[m^2 \left(\frac{\partial\omega}{\partial J}\right)^2 D_J \right]^{1/3}$$

→ orbit decorrelation rate

For lines:

$$1/\ell_\perp^3 \sim m^2 \frac{1}{R^2} \frac{q'^2}{q^2} D_M$$

$$\sim k_\theta^2 \left(\frac{rq'}{q}\right)^2 \left(\frac{1}{R\epsilon}\right)^2 D_M$$

$$\sim k_\theta^2 \left(\frac{\hat{s}}{R\epsilon}\right)^2 D_M.$$

$\hat{s} = \frac{rq'}{q}$ shear parameter

$$\Rightarrow \boxed{1/\ell_\perp \sim \left[k_\theta^2 \left(\frac{\hat{s}}{R\epsilon}\right)^2 D_M \right]^{1/3}}$$

length over which line is decorrelated by B_r scattering by k_θ^{-1} from orbit.

Note: Length scales:

$\Delta_\perp \rightarrow$ scatter scale, radius

$l_{ac} \sim \frac{1}{\Delta |k_\parallel|} \rightarrow$ parallel coherence length of scatterer field

$l_\perp \rightarrow$ decorrelation length for orbit.

then $l_{ac} < l_\perp$ $\Rightarrow$ scatter field must reset before orbit decorrelated

check: $Ku < 1$

$l_{ac}^3 < l_\perp^3$ $\rightarrow$ many kicks.

$$l_{ac}^3 < \frac{1}{k_\theta^2}\frac{(Rq)^2}{\hat{s}^2 D_M}$$

$$Ku \sim \frac{l_{ac}}{\Delta_\perp}\frac{\tilde{B}_r}{B_0} < 1.$$

$$D_M < \frac{(Rq)^2}{\hat{s}^2}\frac{1}{k_\theta^2 \, l_{ac}^3}$$

$$l_{ac}^2\left(\frac{\tilde{B}_r}{B_0}\right)^2 < \frac{(Rq)^2}{\hat{s}^2}\frac{1}{k_\theta^2}\frac{l_{ac}}{l_{ac}^3}$$

$$< \frac{L_s^2}{k_\theta^2}\frac{1}{l_{ac}^2}$$

out

$$\frac{L_s^2}{k_\theta^2 \Delta r^2} \sim \frac{1}{\Delta k_{\parallel}^2} \qquad \tilde{s}/R_\xi = 1/L_s$$

$$\frac{l_{ac}^2}{\Delta_r^2}\left(\frac{\tilde{B}_r}{B_0}\right)^2 < \frac{L_s^2}{k_\theta^2 \Delta^2}\frac{1}{l_{ac}^2} = 1 \checkmark$$

$$\Rightarrow \boxed{l_{ac} < l_k \Leftrightarrow K_{u} < 1}$$

Now:

→ have used mean field theory + hierarchy closure to calculate statistical properties, in chaotic regime, i.e.

→ derived and understood behavior of f → distribution function → of chaotic system.

→ next: – Fokker-Planck approach
– calculation of Lyapunov exponent.