Self-organized criticality

Per Bak, Chao Tang, and Kurt Wiesenfeld
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973
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We show that certain extended dissipative dynamical systems naturally evolve into a critical state, with no characteristic time or length scales. The temporal "fingerprint" of the self-organized critical state is the presence of flicker noise or $1/f$ noise; its spatial signature is the emergence of scale-invariant (fractal) structure.

I. INTRODUCTION

This paper concerns the behavior of spatially extended dynamical systems—that is, systems with both temporal and spatial degrees of freedom. Such systems are common in physics, biology, and even social sciences such as economics. Despite their abundance, there is little understanding of the spatiotemporal evolution of these complex systems.[1] Seemingly disconnected from this problem are two widely occurring phenomena whose very generality require some unifying underlying explanation. The first is a temporal effect known as $1/f$ noise or flicker noise;[2] the second concerns the evolution of a spatial structure with scale-invariant, self-similar (fractal) properties.[3] Here we report the discovery of a general organizing principle governing a class of dissipative coupled systems. Remarkably, the systems evolve naturally toward a *critical* state, with no intrinsic time or length scale. The emergence of the self-organized critical state provides a connection between nonlinear dynamics, the appearance of spatial self-similarity, and $1/f$ noise in a natural and robust way. A short account of some of these results has been published previously.[4]

The usual strategy in physics is to reduce a given problem to one or a few important degrees of freedom. The effect of coupling between the individual degrees of freedom is usually dealt with in a perturbative manner—or in a "mean-field manner" where the surroundings act on a given degree of freedom as an external field—thus again reducing the problem to a one-body one. In dynamics theory one sometimes finds that complicated systems reduce to a few collective degrees of freedom. This "dimensional reduction" has been termed "self-organization," or the so-called "slaving principle,"[5] and much insight into the behavior of dynamical systems has been achieved by studying the behavior of low-dimensional attractors.

On the other hand, it is well known that some dynamical systems act in a more concerted way, where the individual degrees of freedom keep each other in a more or less stable balance, which cannot be described as a "perturbation" of some decoupled state, nor in terms of a few collective degrees of freedom. For instance, ecological systems are organized such that the different species "support" each other in a way which cannot be understood by studying the individual constituents in isolation. The same interdependence of species also makes the ecosystem very susceptible to small changes or "noise." However, the system cannot be too sensitive since then it could not have evolved into its present state in the first place. Owing to this balance we may say that such a system is "critical." We shall see that this qualitative concept of criticality can be put on a firm quantitative basis.

Such critical systems are abundant in nature. We shall see that the dynamics of a critical state has a specific temporal fingerprint, namely "flicker noise," in which the power spectrum $S(f)$ scales as $1/f$ at low frequencies. Flicker noise is characterized by correlations extended over a wide range of time scales, a clear indication of some sort of cooperative effect. Flicker noise has been observed, for example, in the light from quasars,[6] the intensity of sunspots,[7] the current through resistors,[8] the sand flow in an hour glass,[9] the flow of rivers such as the Nile,[2] and even stock exchange price indices.[10] All of these may be considered to be extended dynamical systems. Despite the ubiquity of flicker noise, its origin is not well understood. Indeed, one may say that *because* of its ubiquity, no proposed mechanism to data can lay claim as the single general underlying root of $1/f$ noise. We shall argue that flicker noise is in fact not noise but reflects the intrinsic dynamics of self-organized critical systems. Another signature of criticality is *spatial* self-similarity. It has been pointed out that nature is full of self-similar "fractal" structures, though the physical reason for this is not understood.[11] Most notably, the whole universe is an extended dynamical system where a self-similar cosmic string structure has been claimed.[3] Turbulence is a phenomenon where self-similarity is believed to occur in *both* space and time.

Cooperative critical phenomena are well known in the context of phase transitions in equilibrium statistical mechanics.[12] At the transition point, spatial self-similarity occurs, and the dynamical response function has a characteristic power-law "$1/f$" behavior. (We use quotes because often flicker noise involves frequency spectra with dependence $f^{-\beta}$ with β only roughly equal to 1.0.) Low-dimensional nonequilibrium dynamical systems also undergo phase transitions (bifurcations, mode locking, intermittency, etc.) where the properties of the attractors change. However, the critical point can be reached only by fine tuning a parameter (e.g., temperature), and so may occur only accidentally in nature: It cannot be invoked as an explanation for the dynamical phenomena discussed above where no fine tuning is needed.

In this paper, we argue and demonstrate numerically that dynamical systems with extended spatial degrees of freedom in two or three dimensions naturally evolve into *self-organized* critical states. By self-organized we mean that the system naturally evolves to the state without detailed specification of the initial conditions (i.e., the critical state is an attractor of the dynamics). Moreover, the critical state is robust with respect to variations of parameters, and the presence of quenched randomness. We suggest that this self-organized criticality is the common underlying mechanism behind the phenomena described above.

More specifically, we consider dissipative dynamical systems with local, interacting degrees of freedom. It is essential that the system has many metastable states. Although the details of the local states are not important to the general theory, for the sake of clarity we focus attention on specific models. We choose the simplest possible models rather than wholly realistic and therefore complex models of actual physical systems. Besides our expectation that the overall qualitative features are captured in this way, it is certainly possible that quantitative properties (such as scaling exponents) may apply to more realistic situations, since the system operates at a critical point where universality may apply. The philosophy is analogous to that of equilibrium statistical physics where results are based on Ising models (and Heisenberg models, etc.) which have only the symmetry in common with real systems. Our "Ising models" are discrete cellular automata, which are much simpler to study than continuous partial differential equations.

To illustrate the basic idea of self-organized criticality in a transport system, consider a simple "pile of sand." Suppose we start from scratch and build the pile by randomly adding sand, a grain at a time. The pile will grow, and the slope will increase. Eventually, the slope will reach a critical value (called the "angle of repose"[13]); if more sand is added it will slide off. Alternatively, if we start from a situation where the pile is too steep, the pile will collapse until it reaches the critical state, such that it is just barely stable with respect to further perturbations. The critical state is an attractor for the dynamics. The quantity which exhibits $1/f$ noise is simply the flow of the sand falling off the pile (this is analogous to the situation in an hour glass). One of the models studied in this paper can be thought of as a model of a sand pile, or alternatively as modeling an array of coupled pendula. These models evolve into a critical state: as the pile is built up, the characteristic size of the largest avalanches grows, until at the critical point there are avalanches of all sizes up to the size of the system, analogous to the domain distribution of a magnetic system at a phase transition. The energy is dissipated at all length scales. Once the critical point is reached, the system stays there. The behavior of systems at the self-organized critical point is characterized by a number of critical exponents—which are connected by scaling relations—and the systems obey "finite-size scaling" just as equilibrium statistical systems at the critical point.

The paper is organized as follows. In Sec. II we consider for pedagogical reasons an example in one spatial dimension. In this case the spatial degrees of freedom "decouple" and the system ends up in the *least* stable metastable state. This minimally stable state is a trivial critical state with no spatial patterns and uninteresting temporal behavior. [In a sense this is similar to the one-dimensional (1D) Ising model at zero temperature, or the 1D percolation problem at the percolation threshold.] The interesting cases of two and three dimensions are treated in Sec. III, where simulations on two different "sand-pile automata" are presented. It is shown how one is led naturally to a flicker-noise output spectrum. The connection between the automata and actual physical systems is also discussed. Scaling laws are conjectured, and a finite-size-scaling hypothesis is successfully tested. It is also shown that the criticality is not affected by various types of local randomness; this is essential for the self-organized criticality to be a generic property of naturally occurring dynamical systems. We close with a discussion and summary in Sec. IV. In particular, we discuss the relations between our models and "turbulence." In fact our models can be thought of as "toy" turbulence models where energy is dissipated on all length scales, with spatial correlations described by a generalized Kolmogorov exponent. We also suggest an experiment the reader can perform in his or her own home.

II. THE ONE-DIMENSIONAL CASE AND MINIMAL STABILITY

In this section we study a one-dimensional model of transport. The model has a huge number of metastable states, growing exponentially with the length of the system. To understand the origin of this proliferation of states, or "attractors" of the dynamics of extended systems, consider for instance an array of N uncoupled oscillators, or torsion pendula, each having m stable fixed points. Then the array has m^N stable configurations. The presence of weak coupling does not alter this basic arithmetic. Surprisingly, our model evolves towards the very least stable of all these state. We call this state the global *minimally stable state.*[14] This state is distinctly different from the *critical state* observed in two and three spatial dimensions, and which is the central focus of this paper. Nevertheless, an understanding of the minimally stable state is essential also in understanding the self-organized critical state.

Figure 1(a) shows a model of a one-dimensional sand pile of length N. The boundary conditions are such that sand can leave the system at the right-hand side only. We may think of this arrangement as half of a symmetric sand pile with both ends open. The numbers z_n represent height differences $z_n \equiv h(n) - h(n+1)$ between successive positions along the sand pile. The dynamics is very simple. From the figure one sees that sand is added at the nth position by letting

$$z_n \rightarrow z_n + 1 \ , \qquad z_{n-1} \rightarrow z_{n-1} - 1 \ . \tag{2.1}$$

When the height difference becomes higher than a fixed critical value z_c, one unit of sand tumbles to the lower level, i.e.,

$$z_n \rightarrow z_n - 2 \ ,$$
$$z_{n\pm1} \rightarrow z_{n\pm1} + 1 \quad \text{for } z_n > z_c \ ; \tag{2.2}$$

closed and open boundary conditions are used for the left and right boundaries, respectively,

$$z_0 = 0 \ ;$$
$$z_N \rightarrow z_N - 1 \ ,$$
$$z_{N-1} \rightarrow z_{N-1} + 1 \quad \text{for } z_N > z_c \ .$$

Equation (2.2) is a nonlinear discretized diffusion equation (nonlinear because of the threshold condition). The process continues until all the z_n are below threshold, at which point another grain of sand is added (at a random site) via Eq. (2.1). The model is a *cellular automaton* where the state of the discrete variable z_n at time $t+1$ depends on the state of the variable and its neighbors at time t.

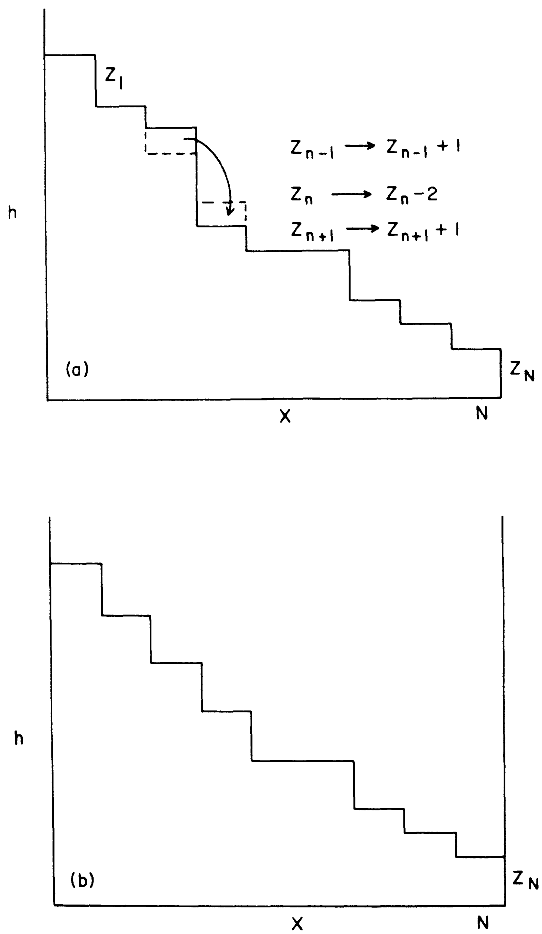

FIG. 1. One-dimensional "sand-pile automaton." The state of the system is specified by an array of integers representing the height difference between neighboring plateaus. (a) There is a wall at the left and sand can exit only at the right; (b) there are walls at both ends.

Alternatively, the system can be thought of as an array of damped pendula in a gravitational field, coupled by torsion springs: The heights $h(n)$ are the winding numbers of the springs, and the z_n are the spring forces on the pendula. When z_n exceeds the critical value z_c, so that the spring force exceeds the gravitational force, the pendulum rotates one revolution, leading precisely to the dynamics Eq. (2.2). By a change of language, this nonlinear diffusion dynamics can also be used to model other of the systems mentioned in the Introduction, describing the flow of electrons, or water, or light, etc. But for convenience we shall continue to use the "sand" language in order to keep in mind a concrete physical picture.

The condition for stability is

$$z_n < z_c \quad (n = 1, 2, \ldots, N) \ ,$$

so the total number of stable states is z_c^N.

If sand is added randomly from an empty system, the pile will build up, eventually reaching the point where *all* the height differences z_n assume the critical value $z_n = z_c$. This is the least stable of all the stationary states. Any additional sand simply falls from site to site (left to right) and falls off at the end $n = N$, leaving the system in the minimally stable state. Alternatively, if one pushes one unit downwards it will also continue its fall until it reaches the edge. In the pendulum picture, this corresponds to kicking one pendulum in the forward direction. This will cause the force on the two nearest-neighbor pendula to exceed the critical value and the perturbation will propagate by a domino effect until it hits the ends of the array. At the end of this process the forces are back to their original values and all pendula have rotated one period. In other words, the effect of a small local perturbation is communicated throughout the system, but the system is robust with respect to noise insofar as it returns to the globally minimally stable state. If units are added randomly, the resulting sandflow is also random white noise, i.e., with power spectrum $1/f^0$. As we shall see in the next section, the robustness of the minimally stable state is lost in two and higher dimensions.

The dynamical selection principle leading to the least stable stationary state is quite independent of how the sand pile is built up. Instead of building the pile by adding sand, we might start with a flat sand surface and slowly raise the left end of the bar. Or, we could randomly add "slope," $z_n \rightarrow z_n + 1$, and let the system obey the dynamics [Eq. (2.2)]. This would represent the dynamics of a system with a random distribution of critical height differences and a uniformly increasing slope. We could also start with a very unstable state, $z_n > z_c$ for all n, and let the system relax. In all these cases the minimally stable state will be reached even if the boundary conditions are such that the sand cannot leave the bar, i.e., closed boundary conditions at both ends [$z_0 = z_N = 0$; Fig. 1(b)].

In the more general case of transport (both in one and higher dimensions), the slope z_n can be thought of as the pressure (or energy, etc.), which builds up precisely to the point where the transport is stationary. A lower slope will prevent transport, and with a higher slope the output will exceed the input for a while until stationarity is restored.

In one dimension, the minimally stable state is critical in the restricted sense that any small perturbation can just propagate infinitely through the system, while any lowering of the slope will prevent this. This is analogous to some other one-dimensional (1D) critical phenomena, such as percolation where at the percolation threshold particles can just percolate to infinity. Also, like other 1D systems, the critical state has no spatial structure, and correlation functions are trivial. In the next section we shall see that in higher dimensions the critical states and their dynamics are dramatically different.

III. SELF-ORGANIZED CRITICALITY IN TWO AND THREE DIMENSIONS

A. Why do we expect self-organized criticality?

The rules (2.1) and (2.2) for the one-dimensional model can easily be generalized to higher dimensions. In two dimensions

$$\begin{aligned} &z(x-1,y)\to z(x-1,y)-1\ , \\ &z(x,y-1)\to z(x,y-1)-1\ , \\ &z(x,y)\to z(x,y)+2\ , \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &z(x,y)\to z(x,y)-4\ , \\ &z(x,y\pm 1)\to z(x,y\pm 1)+1\ , \\ &z(x\pm 1,y)\to z(x\pm 1,y)+1 \quad \text{for } z(x,y)>z_c\ , \end{aligned} \tag{3.2}$$

where we have the square array (x,y), for $1\le x,y\le N$. If one insists on interpreting this discrete diffusion equation in terms of sand piles, the correspondence between the quantity z and the slope is not as straightforward as in the 1D case. The sand columns are represented by the bonds between nearest neighbors in the x and y directions, and $z(x,y)$ represents the average slope in the diagonal direction, i.e., the sum of the height differences in the x and y direction. Equations (3.1) represent the addition of two units at the upper and left bonds. Equations (3.2) represent two units of sand, located at the left and upper bonds at (x,y) sliding in the diagonal direction to the right and lower bonds. [If one wants a more "realistic" sand pile, one can (for instance) define $z(x,y)=2h(x,y)-h(x+1,y)-h(x,y+1)$, in analogy with 1D. When $z>z_c$, one unit of sand slides in the x direction and one in the y direction. The resulting dynamics will involve *next*-nearest-neighbor interactions with the basic physic unchanged; here we stick to the simplest Ising model (3.2).] In principle the slope in two-dimensional (2D) is a vector field, but the scalar field z is easier to work with, and these rules incorporate the essential physics involved. Again, we emphasize that we are interested in the general behavior of nonlinear diffusion dynamics such as Eq. (3.2), and not in sand piles, *per se.*

Naively, one might expect that the situation is the same as in one dimension, namely that the pile will build-up (or collapse) to the minimally stable state where the slopes z_n all assume the critical value. A moments reflection will convince us that it cannot be so. Suppose we punch two units of sand downwards in the diagonal direction by applying rule (3.2). This will render the surrounding sites unstable ($z>z_c$), and the noise will spread to the neighbors, then *their* neighbors, in a chain reaction, *ever amplifying* since the sites are generally connected with more than two minimally stable sites, and the perturbation eventually propagates throughout the entire lattice. The minimally stable state is thus unstable with respect to small fluctuations and cannot represent an attracting fixed point for the dynamics. As the system further evolves, more and more more-than-minimally stable states will be generated, and these states will impede the motion of the noise. *The system will become stable precisely at the point when the network of minimally stable clusters has been broken down to the level where the noise signal cannot be communicated through infinite distances. At this point there will be no length scale, and, consequently, no time scale.* Hence one might expect that the system approaches, through a self-organizing process, a critical state with a power-law correlation function for physically observable quantities, including the power spectrum. In analogy with the discussion for the one dimensional case, the slope (or "pressure") will build up to the point where stationarity is obtained: *this is assured by the self-organized critical state,* but not the minimally stable state. The slope of the critical state is reduced compared to the slope of the minimally stable state.

Suppose that we perturb the critical state locally, by adding one unit, or by locally changing the slope. We expect the perturbation to grow over all length scales. That is, a given perturbation can lead to anything from a shift of a single unit to an avalanche. The lack of a characteristic length scale leads directly to a lack of a characteristic time scale for the fluctuations. As is well known, a random superposition of pulses of a physical quantity with a distribution of lifetimes $D(T)\approx T^{-\alpha}$ (weighed by the average value of the quantity during the pulse) leads to a power frequency spectrum, $S(f)\approx f^{-2+\alpha}$, so a power-law $1/f$ frequency spectrum is equivalent to a power-law distribution of lifetimes.

The nature of the boundary conditions is essential to the nature (though not the *existence*) of the critical state, since the dynamics and the physical situation is largely defined by the properties at the boundaries, for instance whether material is being transport in or out. In this sense the criticality introduced here is different from the criticality at phase transitions where boundary effects always disappear in the thermodynamic limit. We performed simulations with two types of boundary conditions on the 2D cellular automaton: (i) with "closed boundaries" where "sand" cannot leave the box at $x,y=1,N$ and (ii) with open boundary conditions where sand can leave the box on two of the sides, namely at $x=N$ and $y=N$. In the case of closed boundary conditions we also performed simulations in three dimensions and simulations with quenched randomness.

B. Simulations with closed boundary conditions

In the case of closed boundary conditions, $z(0,y)=z(x,0)=z(N+1,y)=z(x,N+1)=0$, the system was simulated with two types of initial condition: far from equilibrium and from a flat surface. In the first case we chose random initial conditions such that z exceeded the threshold z_c at each and every site. This initial state can be visualized as a pile of sand with a huge slope in the diagonal direction. The system then relaxes under the dynamics (3.2) until it reaches a static state. Note that Eq. (3.2) conserves $\sum_n z_n$ except at the boundary, so that any "excess z" must be transported to the boundary for global relaxation to occur. The relaxation is quite a lengthy process. The physical quantity which is transported in this simulation is the "slope."

Once relaxed, the properties of the state are probed by locally perturbing the system. Specifically, we randomly select a minimally stable site [i.e., with $z(x,y)=z_c$] and enforce Eq. (3.2). This corresponds to pushing two units of sand downward in the diagonal direction; we thus induce a "sand slide." This, in general, will cause further slidings as the perturbation spreads. We then measure the total number of slidings s induced by the single perturbation. Note that this operationally defines a domain over which a given perturbation is communicated. After each perturbation, the original static state is restored, and another site is perturbed, and so on. Figure 2 shows a typical domain structure obtained from a number of single-site-induced perturbations. The dark sites are domains affected by perturbing a single interior site. One sees that domains of a variety of sizes exist, from a single site up to one that is comparable to the system size itself. (Note that tickling the interior of a single domain at different individual sites need not result in precisely the same domain boundary.) In a sense, we are measuring the linear response of the system under infinitesimal perturbations. The quantity being measured is the distribution function $D(s)$ of slide sizes. In the simulation, we did not "randomly select" the minimally stable seed site, but systematically went through all of them. We then performed an ensemble average by starting at each of a large number of initial far-from-equilibrium configurations.

Figure 3(a) shows the results for the distribution function $D(s)$, for a 50×50 array, averaged over 200 samples. The log-log plot follows a pretty respectable straight line, with slope -1.0, i.e.,

$$D(s)\approx s^{-\tau}, \quad \tau\approx 1.0 \quad \text{for } D=2 \;. \tag{3.3}$$

That this slope is close to minus one is certainly suggestive, but our current understanding only tells us to expect a power law without nailing down the exponent. Indeed, in three-dimensional (3D) simulations on a $20\times 20\times 20$ array [Fig. 3(b)] one again finds a power law, but now $D(s)=s^{-1.37}$. At small sizes the curve deviates from the straight line because discreteness effects of the lattice come into play.

The fact that the distributions in Fig. 3 begin to deviate from a power law at large cluster sizes is a finite-size effect. For example, in the 50×50 array the deviation begins around $s\approx 200$, whereas simulations on a 20×20 array (not shown) deviate at $s\approx 70$. To verify that this is really a finite-size effect, we borrowed a page from the analysis of equilibrium statistical physics and performed

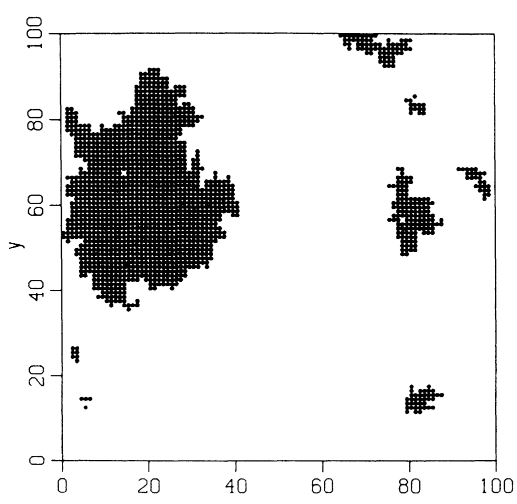

FIG. 2. Typical domain structures resulted from several local perturbations for a 100×100 array. Each cluster is triggered by a single perturbation.

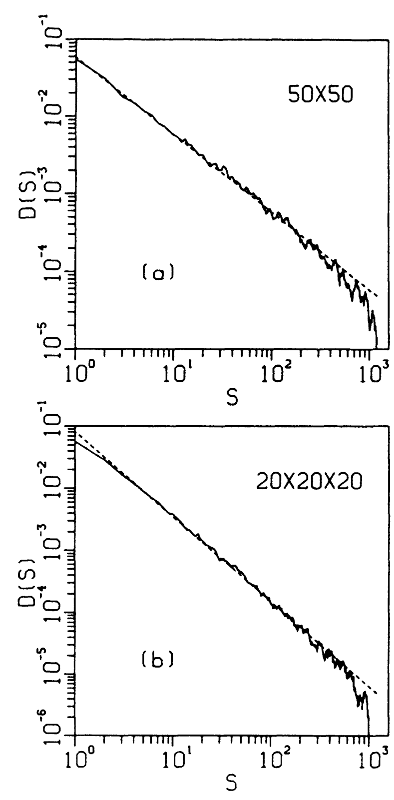

FIG. 3. Distribution of cluster sizes at criticality in two and three dimensions computed as described in the text. The data have been coarse grained. (a) 50×50 array, averaged over 200 samples. The dashed line is a straight line with slope -1.0; (b) $20\times 20\times 20$ array, averaged over 200 samples. The dashed straight line has a slope -1.37.

a finite-size-scaling analysis, a point we will discuss later.

In order to understand the dynamics of the critical state, we now investigate the temporal evolution of the clusters above. Imagine first the effect of a perturbation at a single site on a static critical state. A local perturbation will spread to (some) nearest-neighbor sites, then to next-nearest neighbors, and so on in a "domino" effect, eventually dying out after a total time T, having induced a total of s slidings. In general, T is less than s since the growth rate is usually greater than unity. (We will return to the relation between growth rate and cluster distribution at a later stage.) Figure 4 shows the distribution of lifetimes $D(T)$ weighted by the average response s/T. This quantity also has power-law behavior,

$$D(T) \approx T^{-\alpha} , \quad \alpha \approx 0.43 \text{ for } D=2, \quad \alpha \approx 0.92 \text{ for } D=3 . \tag{3.4}$$

Note that the curves for the lifetime distribution fit a power law only over a decade or so, while the cluster size distributions fit for at least two decades. This is due to the fact that the lifetime of a cluster is much smaller than its size, thus limiting the range over which we have reliable data in Fig. 4.

We now give the "sliding" a new meaning, namely a point of energy dissipation: When a sliding event occurs, a unit of energy is dissipated. Let us introduce the quantity $f(\mathbf{x},t)$ representing dissipation of energy at site $\mathbf{x}$ at time t. The process (3.2) represents dissipation of one unit of energy at the position $\mathbf{x}$ at time t, $f(\mathbf{x},t)=\delta(\mathbf{x},t)$. Thus, the size s of a cluster represents the total energy dissipated as a consequence of a local disturbance. The total number of slidings (3.2) at time t (the growth rate of the cluster) represents the (instantaneous) dissipation rate, $F(t)=\int f(\mathbf{x},t)dx$. The total cluster size $s=\int F(t)dt$, integrated over the duration of a single avalanche.

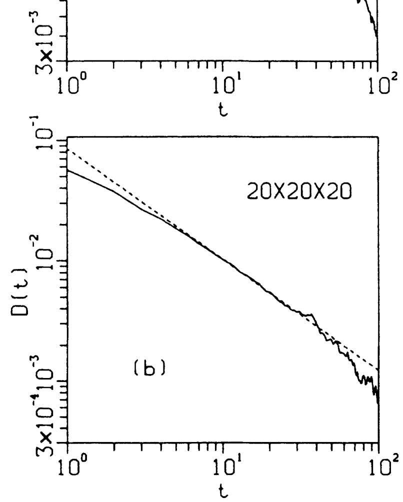

FIG. 4. Distribution of lifetimes corresponding to Fig. 3. (a) For the 50×50 array, the exponent $\alpha \approx 0.43$ yields a $1/f$ noise spectrum $f^{-1.57}$; (b) 20×20×20 array, $\alpha \approx 0.92$, yielding an $f^{-1.08}$ spectrum.

We now consider the response to a situation where the system is locally perturbed randomly in space and time, so that the dissipation $F(t)$ is a superposition of the events above, acting concurrently and independently. We want to calculate the power spectrum $S(f)$ of the quantity $F(t)$ defined by

$$S(f)=\int \langle F(t_0+t)F(t_0)\rangle \exp(2\pi i f t)dt , \tag{3.5}$$

where $\langle\ \rangle$ represents an average over all times t_0. In fact, the power-law distribution of lifetimes, Eq. (3.4), leads to a power law for $S(f)$, as we now describe.

The idea that a distribution of relaxation times T, operating simultaneously and independently, can lead to $1/f$ noise is an old one, originally due to van der Ziel.[13] The present context requires a small generalization of the argument, leading to the following simple formula for the spectrum expressed in terms of the *weighted* distribution function:

$$S(f)=\int^{1/f} TD(T)dT , \tag{3.6a}$$

to be compared with van der Ziel's formula for exponential relaxation

$$S(f)=\int TD'(T)/[1+(fT)^2]dT \approx f^{-2+\alpha} , \tag{3.6b}$$

where $D' \approx T^{-\alpha}$ is the distribution of *linear* relaxation times, each having the same intensity. To see how these last two equations are related, consider first the relaxation due to a single event in a given domain. In general, the resulting correlation function $c(t)$ is characterized by two parameters: the duration of the response T and the total integrated response $\int c(t)dt \equiv A$. (In fact, in our simulations, A is just the total slidings s.) For example, for a linear process, $c(t)=(A/T)\exp(-t/T)$, giving a contribution $S_{A,T}(f)$ to the power spectrum of (up to a numerical constant)

$$S_{A,T}(f)=A/(1+4\pi f^2T^2) .$$

The total power spectrum is the (incoherent) sum of these over all events. Now, the usual van der Ziel argument assumes that A is strictly proportional to T for all events (i.e., that each event has the same initial effect), which leads directly to Eq. (3.6b). More generally, A and T need not be strictly proportional [see Eq. (3.9)]. In our simulations, since A is simply the total energy dissipated s, we get

$$S(f)=\sum S_{A,T}(f) \approx \int dT\, T[sD'(T)/T]/(1+4\pi f^2T^2) ,$$

where the quantity in square brackets is just what we called $D(T)\equiv$distribution of lifetimes "weighted by the average response s/T." Finally, treating the denominator as a high-frequency cutoff, we recover the simple expression Eq. (3.6a).

Thus, the power-law distribution for lifetimes leads immediately to the $1/f$ spectrum

$$S(f) \approx f^{-\beta} = f^{-2+\alpha} ,$$
$$\beta \approx 1.57 \text{ for } D=2, \quad \beta \approx 1.08 \text{ for } D=3 . \tag{3.7}$$

Hence, our prediction that the system must go to a critical state with power-law spatial correlations and $1/f$ noise is fully consistent with the numerical simulations. The $1/f$ noise is the temporal signature of the self-similar properties of the critical state.

The critical state can be approached in a different way, which shows directly how the $1/f$ noise represents the intrinsic dynamics in a stationary dynamic system in the self-organized critical state. Starting from a flat surface, $z_n = 0$, the slope or pressure is increased by one unit at a random position (x,y),

$$z(x,y) \rightarrow z(x,y)+1 . \tag{3.8}$$

Then z is increased by one at another position n, and so on. When z eventually exceeds the critical value z_c somewhere, the system evolves according to (3.2) until it becomes stable again, and a cluster involving s slidings is created. After a while the system arrives at a (statistically) stationary state with clusters of all sizes up to the size of the system. The system "self-averages" over many configurations as time progresses, and no resetting is necessary. The process described here simulates a situation where the slope increases gradually and takes the system to the critical point. This has a lot of similarity with turbulence where energy is fed into the system in a long-wavelength mode. We find, as before, that energy is dissipated on all length scales and all time scales. We shall return to the subject of turbulence in the discussion section.

Figure 5 shows the distribution of cluster sizes measured in a 2D system of size 50×50 after the system arrived at the critical state. The distribution is indistinguishable from the one in Fig. 3 obtained from the relaxed state, indicating that we are indeed dealing with the same critical state. The distribution of lifetimes was also measured and was indistinguishable from that of Fig. 4. Similar measurements were also performed in a 3D system of size $20 \times 20 \times 20$ and again the results were indistinguishable from their counterparts in Figs. 3 and 4. Thus, in a situation where the processes occur independently, the system exhibits $1/f$ noise.

In principle, in an actual physical situation, the build-up $z(x,y) \rightarrow z(x,y)+1$ might take place at a speed which is so large that the individual processes will overlap, and the system will go into a state with higher-than-critical slope. This will cause a cutoff in the time scales over which $1/f$ noise occurs. This represents the situation for a wildly turbulent system. Thus, for $1/f$ noise to occur over many orders of magnitude, it is important that the time scale of the build-up of the critical state is much larger than the time scales of the measured noise. To summarize, the $1/f$ noise may be cut off either by finite-size effects, or by too rapid flow through the stationary state.

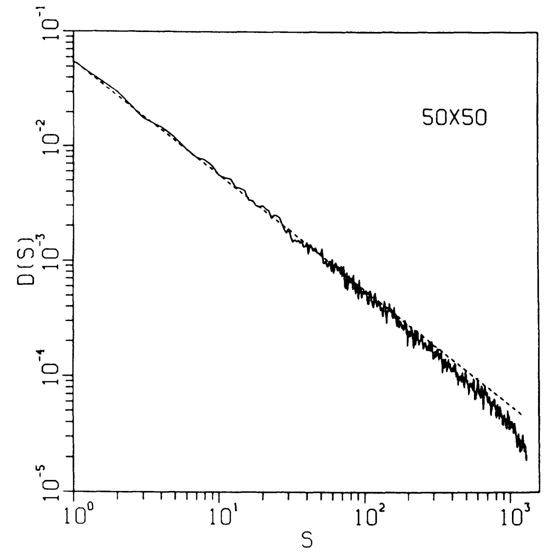

FIG. 5. Cluster size distribution for system built up from scratch according to rules (3.2) and (3.8) for a 50×50 array. The curve is indistinguishable from that in Fig. 3(a). For this system the system is in a stationary critical state and it is self-averaging. Rule (3.8) has been applied 100 000 times to the stationary critical state to obtain this curve.

For a picturesque example, consider a mountain landscape built from some long-wavelength tectonic plate motion. When the landscape reaches the critical state, the landscape will be self-similar, and there will be avalanches on "all time scales." Clearly, the geological time scales involved in building the landscape separate from all realistic avalanche lifetimes.

The fact that $1/f$ noise arises from the random superposition of events can be demonstrated directly. Figure 6 shows the total dissipation as a function of time $F(t)$ formed by superimposing the time dissipation functions $F_{\rm cl}(t)$ for the evolution of the individual clusters in the $20 \times 20 \times 20$ system, starting each cluster at a random time. The curve has the features of a $1/f$ noise, i.e., there are events on all time scales. It is more regular than white noise, and less regular than a random walk. Figure 7 shows the power spectrum $S(f)$ of the curve. Indeed, the log-log plot shows a power-law behavior with the exponent $\beta \approx 0.98$ as expected from the distribution of weighted lifetimes. The crossover to white noise at $f \approx \frac{1}{50}$ is the same finite-size effect as found for the distribution of lifetimes. The absence of very large clusters reduces correlations at very low frequencies.

How robust are these results? It is important that our results be insensitive to randomness to have any chance to explain $1/f$ noise in general, since few of the systems where $1/f$ noise occurs are simple translational invariant systems. To test this, we modified our model by introducing some "quenched randomness." Specifically, we removed certain nearest-neighbor connections in the array, the disrupted connections being fixed through the

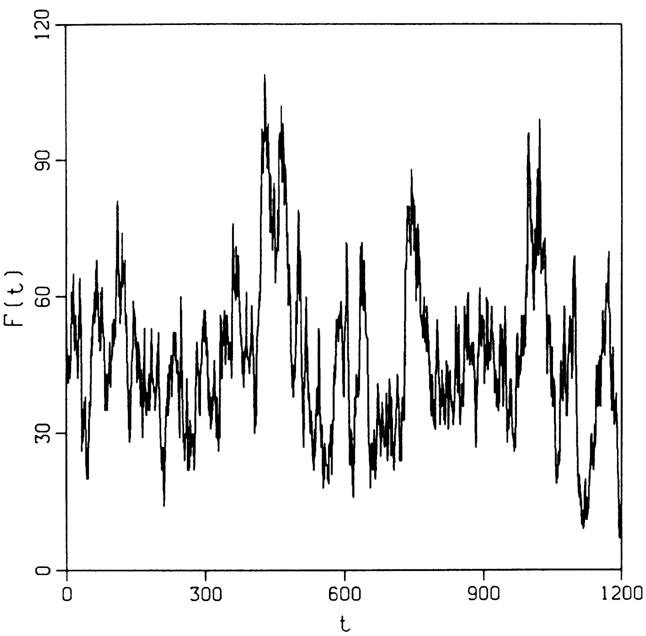

FIG. 6. $F(t)$ generated by superimposing randomly the response represented by the individual clusters for a $20\times20\times20$ array. Note the fluctuations on a wide range of time scales.

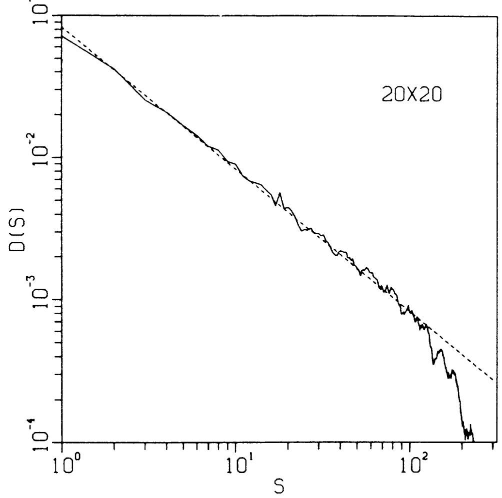

FIG. 8. Cluster size distribution for 20×20 array with 10% of the bonds removed, averaged over 1000 samples. The slope is -1.0, same as for the pure system.

simulation. We found that removing at random as many at 25% of the bonds still led to power-law distributions. Figure 8 shows the cluster size distribution for a situation with 10% of the bonds removed. No change in the exponent was detected. Whether this is due to true universality of the exponent, or due to our inability to detect a small change numerically is not known. Quantitatively, the presence of this kind of randomness increases the mean value of the pressure z since fewer nearest neighbors are available to communicate the noise. In the sand picture, the introduction of impedements causes a build-up to a higher slope, by the self-organization process, until criticality is achieved again.

The exponents τ and α representing the spatial and temporal evolution of the clusters, respectively, can be related through "scaling relations." If the perturbation grows with an exponent γ within the clusters, the lifetime T of a cluster is related to its size by

$$s\approx T^{1+\gamma} . \tag{3.9}$$

The distribution of lifetimes weighted by the average response $D(T)$ can then be related to the distribution of cluster sizes

$$D(T)=(s/T)D[s(T)](ds/dT)$$
$$\approx T^{-(\gamma+1)\tau+2\gamma}\equiv T^{-\alpha} . \tag{3.10}$$

The scaling laws

$$\alpha=2-\beta=(\gamma+1)\tau-2\gamma \tag{3.11}$$

can be directly read off Eqs. (3.7) and (3.10). Since we have measured α and τ independently, we can find the anomalous growth exponent γ from Eq. (3.11),

$$\gamma\approx0.57 \text{ for } D=2, \quad \gamma\approx0.71 \text{ for } D=3 . \tag{3.9'}$$

C. Simulations with open boundaries

In this set of simulations we consider an "open-ended" system where particles are transported through the system and are allowed to leave the box at the two edges $x=N$ and $y=N$. The idea is to simulate a situation resembling some of the systems known to have $1/f$ noise. In the hour glass, sand is transported through the system; in quasars (and in the case of sunspots) light is transported through an open-ended system, and in the case of rivers, water is transported.[14] The open-end configuration is simulated by modifying the first equation of (3.2) at the boundaries

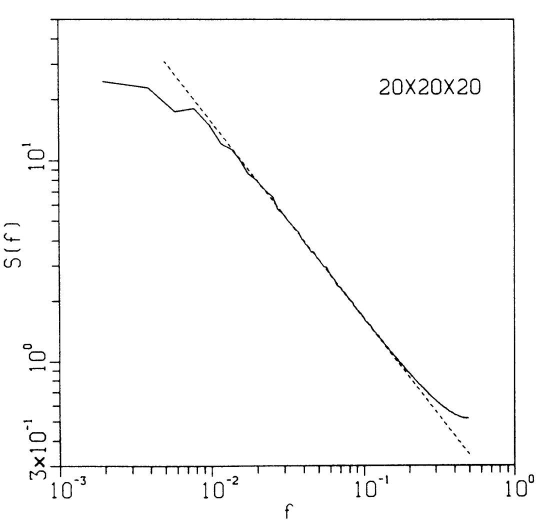

FIG. 7. Power spectrum of the function $F(t)$ depicted in Fig. 6. The spectrum is $1/f$, varying as $f^{-0.98}$. The crossover to white noise at very low frequencies is a finite-size effect.

$$z(N,y) \to z(N,y)-3\ ,$$
$$z(x,N) \to z(x,N)-3\ , \qquad (3.12)$$
$$z(N,N) \to z(N,N)-2\ ,$$

when the appropriate value of z exceeds the critical value. Starting from scratch, the sand pile is built up by adding particles randomly according to the rule (3.1). After each particle is added, the system is allowed to relax (if need be) according to (3.2). The relaxation represents the sliding in the diagonal direction of one particle. In addition to the boundary conditions (3.12) we also studied a diagonally cut system where particles are allowed to leave the pile along the edge $x+y=N$, perpendicular to the flow direction. The results for the two systems were the same. Again, the process is continuous and no resetting is necessary. After an initial transient period, we found that the system reached a statistically stationary state with clusters of all sizes.

Figure 9 shows the critical sandpile in the diagonal case. The heights represent the value of the bonds as described above. For each cluster, we monitor only the flow $f(t)$ of sand that falls off the edges of the box. Most of the time, the relaxation following a single kick results in no sand falling off the edge, though, of course, on average one unit of sand falls off for every unit added, once steady state has been achieved. For those perturbations which do result in a response, we again form $D(T)$ as before. Figure 10 shows $D(T)$ for a 2D system of size 75×75. Again the distribution follows a power law for a decade or so, this time with exponent equal to unity, $\alpha\approx1.05$. The distribution on lifetimes translates directly into a power-law frequency spectrum

$$S(f)\approx f^{-\beta},\quad \beta\approx0.95\ . \qquad (3.13)$$

(To convince himself that this description has some connection with reality, we suggest that the reader study visually the fluctuating flow of an hour glass and notes the pulses of widely different time scales.) We cannot rule out that the exponent is identical to unity with the present numerical accuracy.

To test the assertion that the lifetime distribution is scaling, with cutoff due only to the finite size, we applied a technique familiar from the study of critical phenomena

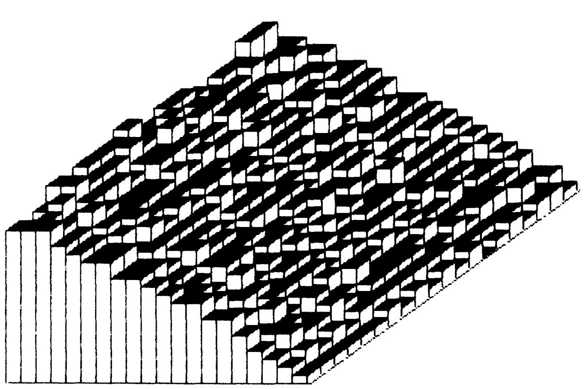

FIG. 9. The critical sand pile for a 2D system with open edge perpendicular to flow.

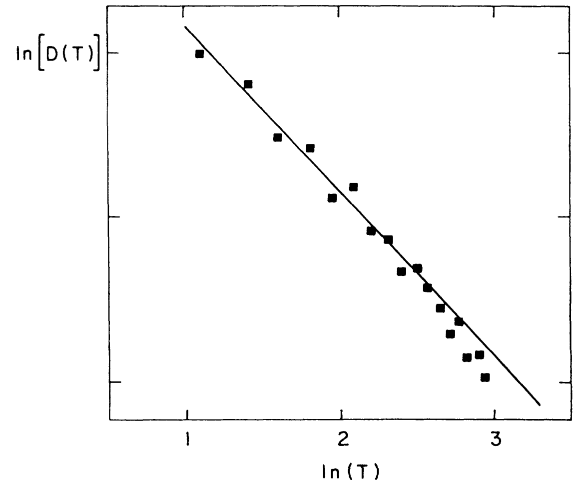

FIG. 10. Lifetime distribution of the sand flow for a system of size 75×75. The straight line has a slope -1.05.

of second-order transitions. At a critical point one expects $D(T)$ to obey a finite-size-scaling relation

$$D(T)=T^{-\alpha}F(L^{\sigma}/T)\ , \qquad (3.14)$$

where F is a crossover scaling function of the single scaled variable L^{σ}/T, and σ is a dynamical critical exponent. Similarly, one would expect a scaling relation for the size s of bulk clusters to obey a relation of the form

$$D(s)=s^{-\tau}F'(L^{d}/s)\ , \qquad (3.15)$$

where the critical exponent d can be thought of as the fractal dimension of the clusters. Figure 11 shows the product $D(T)T^{\alpha}$ versus L^{σ}/T for $\sigma=0.75$: The points for different L all fall on the same curve $F(x)$ to within numerical accuracy. (Actually it would not be surprising if the scaling was anisotropic—because of the anisotropic boundary conditions—such that F would be a function of two variables. The exponent found here would then be the smaller of the two exponents.) The finite-size-scaling

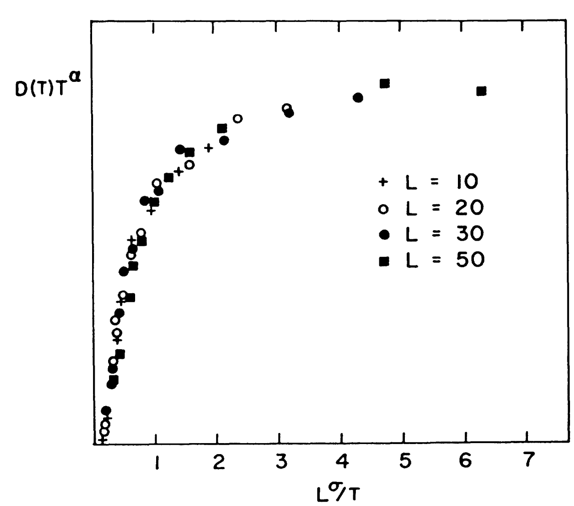

FIG. 11. Finite-size-scaling plot.

conjecture (3.15) was also successfully tested for the closed boundary case; in this case the scaling cannot be anisotropic since the dynamics is isotropic. An exponent of $d \approx 1.92$ was found for the 2D system. We suggest that experiments be performed to check the finite-size-scaling conjecture. This would be a direct proof of spatial organization.

IV. SUMMARY AND DISCUSSION

To summarize, our general arguments and numerical simulations show that dissipative dynamical systems with extended degrees of freedom can evolve towards a self-organized critical state, with spatial and temporal power-law scaling behavior. The spatial scaling leads to self-similar "fractal" structure. The frequency spectrum is $1/f$ noise or flicker noise with a power-law spectrum $S(f) \approx f^{-\beta}$.

Thus, in our picture $1/f$ noise is not noise but reflects the generic dynamics of extended dynamical systems. We found values of β tolerably close to one (and certainly between 0 and 2). It remains to be seen to what extend systems can be grouped into universality classes within which the exponents are the same, depending on symmetry, dimension, and so on. We strongly suspect that the criticality discovered here cannot depend on the local details of the models, in analogy with equilibrium second-order phase transitions.

Moreover, we conclude that $1/f$ noise is intimately related to the underlying spatial organization. This can be tested directly, for instance by measuring the frequency cutoff versus the system size. In retrospect, it is hard to see how $1/f$ noise, with long temporal correlations, could possibly occur without long-range spatial correlations, except by "fine-tuning" models with few degrees of freedom.

We believe that the concept of self-organized criticality can be taken much further and might be *the* underlying concept for temporal and spatial scaling in dissipative nonequilibrium systems. One of our models (with closed boundary conditions) could be considered a toy model of generalized turbulence, with dissipation correlated on all length scales. Of course, there is no direct connection with (for instance) the Navier-Stokes equation, where the metastability is due to the storage of kinetic energy in vortices, not potential energy as in the models discussed here. Nevertheless, there is a one-to-one connection for the phenomenology used to describe the two situations, and our simulations may provide an entrance into the problem of dynamical scale invariance. For instance, we could measure equal time correlation functions for the dissipation, $S(x) = \langle\langle f(\mathbf{x}_0+\mathbf{x},t) f(\mathbf{x}_0,t)\rangle\rangle$, where $\langle\langle\ \rangle\rangle$ denotes a spatial average. Because of the scaling properties of the critical state, we expect

$$S(x) \approx x^{-\mu} ,$$

with μ the "Kolmogorov exponent" for our "Ising" turbulence model. This equation indicates that the dissipation at any given instant takes place on a fractal rather than in the bulk. It would be interesting to calculate this exponent and compare with experiments on real turbulence.

Another application might be to the problem of the dynamics of quenched glasses and spin glasses. These are frustrated systems far from equilibrium, and it appears that they have much in common with our three-dimensional (3D) model. It might be that glasses quench into critical states where they are just barely stable with respect to noise, but this speculation has to be developed further. In fact if one wants to understand $1/f$ noise in resistors one probably first has to understand the "glassy" dynamics of the material forming the resistor.

Finally, we invite the reader to perform the following home experiment. To demonstrate self-organized criticality, one needs a shoebox and a cup or two of sand—sugar or salt will do in a pinch. Wet the sand with a small amount of water, mix, and gather the sand into the steepest possible pile in one corner of the box. The angle of repose (i.e., the threshold slope) is larger for wet sand, so as the water evaporates, one observes a sequence of slides—some very small, others quite large—occurring at random places on the pile. (The evaporation process can be sped up by placing the box on a warm surface, or under direct sunlight.) This is essentially the situation analogous to that of Fig. 5 where the slope of the pile is increased gradually, but instead it is the parameter z_c that is gradually *decreasing*. This experiment is also exceptionally portable, and is best done on a sunny day at the beach.

ACKNOWLEDGMENT

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[15]In some realistic situations the quantity which is measured is not the flow of the quantity which is being transported. For instance, in resistors the fluctuations are probably caused by the dynamics of the bulk material, not by the current itself. In other situations the experiment directly measures the energy release (photons or phonons) related to the individual processes in the bulk. These scenarios are more like the "closed boundary" case.

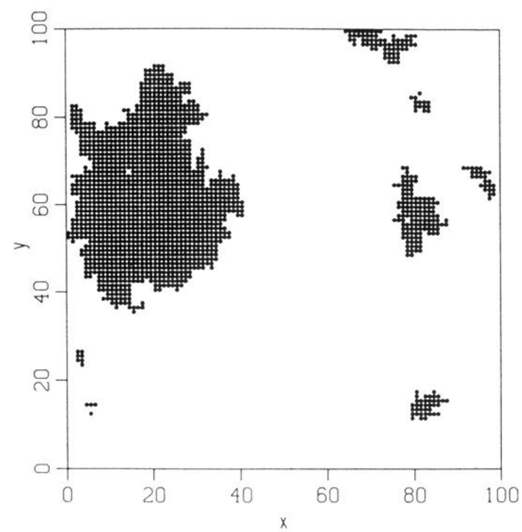

FIG. 2. Typical domain structures resulted from several local perturbations for a 100×100 array. Each cluster is triggered by a single perturbation.