$1/f$ noise and self-organized criticality

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1 Introduction

Until now we have explored the intermittent problem, from which the multiplicative process associated with log-normal distribution of probability, time series and Levy flights theory as well as Pareto (inverse fractional power) distribution, etc have been studied systematically. But $1/f$ "noise", which is one of the important results of these theories has been left. By definition, $1/f$ noise is a signal or process with a frequency spectrum such that the power spectral density (energy or power per Hz) is inversely proportional to the frequency of the signal:

$$S(f) \propto \frac{1}{f^{\alpha}} \tag{1}$$

where f is the frequency and $0 < \alpha < 2$, with exponent α usually close to 1. For instance, the statistics of the flooding of the Nile has been a popular example $1/f$ noise.

Actually, we will show in this note that the $1/f$ noise is not noise but the temporal signature of the self-similar properties of the critical state of self-organized critical systems, whose spatial self-similarity results in the "fractal" structure that has been given in the former notes. Turbulence is a phenomenon where self-similarity is believed to occur in both space and time. So it plays an import role in our study of plasma. For example, M. Endler indicated that a probability distribution function of turbulence induced fluxes shows a long tail with 10% of the largest flux events being responsible for 50% of the transport [1]. The common feature for thid kind of systems is that the power-law temporal or spatial correlations extend over several decades where naively one might suspect that the physics would vary dramatically. This stimulates us to learn $1/f$ distribution and $1/f$ noise.

In this note, we will investigate the general theory for $1/f$ law, i.e. the self-organized criticality in which $1/f$ noise or flicker noise can be identified with the dynamics of the critical state. We will study the flow of sand in models of "running" sandpiles. In this model, at very short time scales, the flow is dominated by single avalanches. These avalanches overlap at intermediate time scales; their interactions lead to $1/f$ noise in the flow. We will also introduce the conservation laws, showing that scaling in this region is a consequence of them.

2 $1/f$ noise from $1/f$ distribution

We have observed the $1/f$ distribution in the former notes, for instance the log-normal distribution with a large dispersion can be mimicked by a 1/x distribution over a wide range of x [2]. This because the probability that the variable $x/\bar{x}$ lies in the interval $d(x/\bar{x})$ is

$$g(x/\bar{x})d(x/\bar{x}) = \frac{exp\left[-(log\, x/\bar{x})^2/2\sigma^2\right]}{(2\pi\sigma^2)^{1/2}} \frac{d(x/\bar{x})}{x/\bar{x}} \tag{2}$$

where $\bar{x}$ is the mean and σ^2 is the square of the dispersion of the distribution. Let $f = x/\bar{x}$, yielding the $log\ g$as a function $log\ x$ like

$$log\ g(f) = -log\ f - \frac{1}{2}\left[(log\ f)/\sigma\right]^2 - \frac{1}{2}log(2\pi\sigma^2). \tag{3}$$

Obviously, the distribution $g(f)$ is to be $1/f$ for the case $\sigma \to \infty$ as the last term is a constant and negligible. Given $\sigma^2 = N\bar{\sigma}^2$ for multiplicative processes, which can be described by the log normal distribution under mild conditions, the greater the value of N, the greater the number of e-folds or decades over which the distribution function mimics $1/f$ distribution.

So the next question is how to get the $1/f$ noise from such power law distribution. That such a **scale-invariant** distribution function of relaxation times $[\rho(\tau)d\tau = d\tau/\tau]$ of random events leads to a $1/f$ noise spectrum was first pointed out by van der Ziel [3] and extended by Machlup. A purely random process generally has an autocorrelation of the form $\langle\tilde{\phi}(t_1)\tilde{\phi}(t_2)\rangle = \left|\tilde{\phi}(0)\right|^2 e^{-t/\tau}$. The power spectrum of such a process has the Lorentz shape by Fourier transform:

$$S(\omega) \propto \tau/(1+\omega^2\tau^2).$$

If we have a large collection of different random processes, each with its own correlation time τ, then the power spectrum of the whole ensemble depends on the statistical distribution $\rho(\tau)$ of these correlation times. If These processes have not been filtered, then our conjecture is that the weighting function is *scale invariant*:

$$\rho(\tau)d\tau \propto d\tau/\tau.$$

This gives a power spectrum

$$\int_{\tau_1}^{\tau_2} S_\tau(\omega)\rho(\tau)d\tau \propto \int_{\tau_1}^{\tau_2} \frac{\tau}{1+\omega^2\tau^2}\frac{d\tau}{\tau} = \left.\frac{tan^{-1}\omega\tau}{\omega}\right|_{\tau_1}^{\tau_2}. \tag{4}$$

If the scale invariance extends over many orders of magnitude, i.e., if τ_2/τ_1 is a large ratio, then the spectrum is $1/\omega$ over a correspondingly large range.

We propose that if, in a system of interest, the distribution of relaxation times is determined by a multiplicative process, then that distribution becomes log normal. However, as discussed above, for a considerable range, a log normal $\rho(\tau)$ is mimicked by $1/\tau$, as required by derivation of $1/\omega$ above. For example, the Nile has been observed to exhibit $1/f$ noise. We would explain this by considering the many stages through which a drop of rain at the source of a river must successfully pass to reach the mouth of the river. First, atmospheric conditions must lead to rain to create the drop and then wind, temperature, and ground porosity conditions must allow the drop to continue downstream at each stage of the river. The resulting log-normal distribution for the flow rate yields, in a natural fashion, a $1/f$ noise in the river level at the mouth.

3 Self-organized criticality

As we discussed above, there are similarly diverse temporal processes generically exhibiting $1/f$ noise. These phenomena lack natural length and time scales and instead possess scale-invariant or self-similar features. The concept of fractals has been successful in characterizing the geometrical aspects of scale-invariant systems, while methods developed from the studies of critical phenomena may provide the necessary analytical tools.

The concept of "self-organized criticality" (SOC) was proposed by Bak, Tang, Wiesenfeld(BTW) [4] [5] as an explanation for the behaviour of a cellular-automata model (sandpile model) they developed. By definition, SOC is a property of (classes of) dynamical systems that have a critical point as an attractor. Their macroscopic behaviour thus displays the spatial and/or temporal scale-invariance characteristic of the critical point of a phase transition, but without the need to tune control parameters to precise values. It provides a connection between nonlinear dynamics, the appearance of spatial self-similarity, and $1/f$ noise in a natural and robust way. Here, the self-organized means that dynamical systems with extended spatial degrees of freedom in two or three dimensions, numerically speaking, naturally evolves to the state without detailed specification of the initial conditions (i.e., the critical state is an attractor of the dynamics). So the critical state in this theory is different from that in static critical phenomena, in which scale invariance and self-similarity are only exhibited at a few isolated, or critical, points in the parameter space. For example, the Ferromagnet, described by Ginsburg/Landau theory

$$\frac{\partial n}{\partial t} = c\nabla^2\eta - a'(T - T_c)\eta - b'\eta^3$$

has a critical temperature T_c, so it's a critical-point problem. However SOC describes the systems exhibit self-similarity without any tuning of parameters. The authors suggest that this self-organized criticality is the common underlying mechanism behind the phenomena described above. To illustrate the basic idea of self-organized criticality in a transport system, we will consider the sandpile model as well as Schelling Segregation Model in the following parts.

3.1 Characteristics of the SOC

SOC was suggested to be the typical behavior of interacting many-body systems. BTW claimed that, under very general conditions, dynamical systems organize themselves into a state with a complex but rather general structure. The systems are complex in the sense that no single characteristic event size exists: there is not just one time and one length scale that controls the temporal evolution of these systems. Although the dynamical response of the systems is complex, the simplifying aspect is that the statistical properties are described by simple power laws. What's more, the claim by BTW was that this typical behavior develops without any significant "tuning" of the system from the outside.

Phenomena in very diverse fields of science have been claimed to exhibit SOC behavior, such as sandpiles, earthquakes, forest fires, electric breakdown, motion of magnetic flux lines in superconductors, dynamics of magnetic domains and growing interfaces. However there does not exist a clear-cut and generally accepted definition of what SOC is [6]. Nor does a very clear picture exist of the necessary conditions under which SOC arises. Then what kind of systems will evolve into a SOC dynamical state? A separation of time scales is required. The process connected with the external driving of the system needs to be much slower than the internal relaxation processes. The separation of time scales is intimately connected with the existence of *thresholds* and *metastability*. The strong drive will not allow relaxation from one configuration (metastability) to another.

Then the question becomes where SOC is to be found. Certainly, we will expect SOC behavior in *slowly driven*, *interaction-dominated threshold* (SDIDT) systems. SDIDT focuses on the two unique feature of such system:the interesting behavior arises because many degrees of freedom are interacting; and the dynamics of the system must be dominated by the mutual interaction between these degrees of freedom, rather than by the intrinsic dynamics of the individual degrees of freedom. If a system exhibits power laws without any apparent tuning then it is said to exhibit self-organized criticality; SOC is a *phenomenological* definition rather than a constructive one.

3.2 Sandpile model

The experiments on sandpile are a prime example of SOC. So we will introduce SOC using sandpile model both in 1D and 2D.

3.2.1 Description of sandpile model using 1D case

Fig.1 shows a model of a one-dimensional sand pile of length N. The boundary conditions are such that sand can leave the system at the right-hand side only. We may think of this arrangement as half of a symmetric sand pile with both ends open. The numbers z_n represent height differences $z_n = h(n) - h(n+1)$ between successive positions along the sand pile. The dynamics is very simple. From the figure one sees that sand is added at the nth position by letting

$$\begin{aligned} z_n &\to z_n + 1, \\ z_{n-1} &\to z_{n-1} - 1. \end{aligned} \tag{5}$$

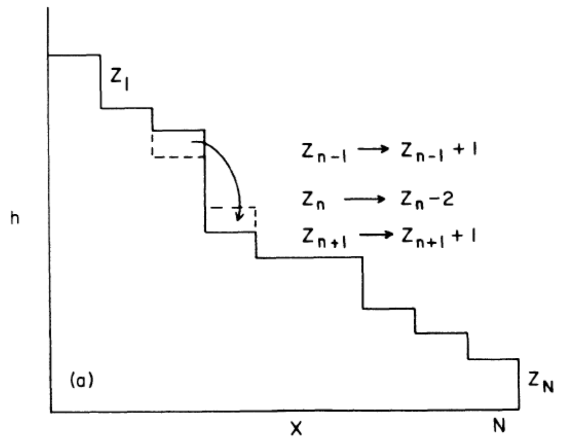

Figure 1: One-dimensional "sand-pile automaton".

Suppose we start from scratch and build the pile by randomly adding sand, a grain at a time. The pile will grow, and the slope will increase. Eventually, the slope will reach a critical value (i.e., the critical value of height difference z_c); if more sand is added it will slide off, i.e., one unit of sand tumbles to the lower level

$$\begin{aligned} z_n &\to z_n - 2, \\ z_{n\pm1} &\to z_{n\pm1} + 1. \end{aligned} \tag{6}$$

Alternatively, if we start from a situation where the pile is too steep, the pile will collapse until it reaches the critical state, such that it is just barely stable with respect to further perturbations. The critical state is an attractor for the dynamics. The quantity which exhibits $1/f$ noise is simply the flow of the sand falling off the pile. The model is a *cellular automaton* where the state of the discrete variable z_n at time $t+1$ depends on the state of the variable and its neighbors at time t.

In this model the effect of a small local perturbation is communicated throughout the system via the nearest-neighboring law. In the more general case of transport (both in one and higher dimensions), the slope z_n can be thought of as the pressure (or energy, etc.), which builds up precisely to the point where the transport is stationary. A lower slope will prevent transport, and with a higher slope the output will exceed the input for a while until stationarity is restored.

In one dimension, the minimally stable state is critical in the restricted sense that any small perturbation can just propagate infinitely through the system, while any lowering of the slope will prevent this. This is analogous to some other one-dimensional critical phenomena, such as percolation where at the percolation threshold particles can just percolate to infinity. Also, like other 1D systems, the critical state has no spatial structure, and correlation functions are trivial.

3.2.2 two dimensions and power laws

For 2D case, there is a square grid of boxes and at each time step a particle is dropped into a randomly selected box. When a box accumulates four particles, they are redistributed to the four adjacent boxes via

$$\begin{aligned} z(x,y) &\to z(x,y) - 4, \\ z(x \pm 1, y) &\to z(x \pm 1, y) - 1, \\ z(x, y \pm 1) &\to z(x, y \pm 1) - 1, \end{aligned} \tag{7}$$

where $1 \leq x, y \leq N$ like 1D case. Redistributions can lead to further instabilities and avalanches of particles in which many particles may be lost from the edges of the grid. One measure is given by the number of boxes that participate in the redistributions.

The situation here is not like 1D case. One might expect that the system approaches, through a self-organizing process, a critical state with a power-law correlation function for physically observable quantities, including the power spectrum. In analogy with the discussion for the one dimensional case, the slope (or "pressure") will build up to the point where stationarity is obtained: $this\ is\ assured\ by\ the\ self$ $organized\ critical\ state$, but not the minimally stable state.

Suppose that we perturb the critical state locally, by adding one unit, or by locally changing the slope. We expect the perturbation to grow over all length scales. That is, a given perturbation can lead to anything from a shift of a single unit to an avalanche. The lack of a characteristic length scale leads directly to a lack of a characteristic time scale for the fluctuations. The physical quantity which is transported in this model is the "slope."

We then measure the total number of slidings s induced by the single perturbation. Note that this operationally defines a domain over which a given perturbation is communicated. After each perturbation, the original static state is restored, and another site is perturbed, and so on. Fig.2 shows a typical domain structure obtained from a number of $single-site-induced$ perturbations. The dark sites are domains affected by perturbing a single interior site. One sees that domains of a variety of sizes exist, from a single site up to one that is comparable to the system size itself.

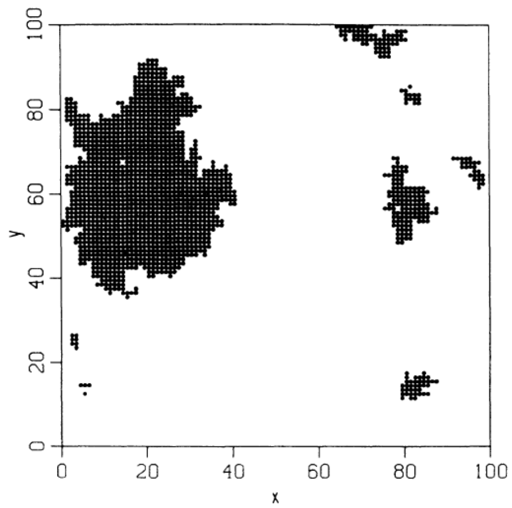

Figure 2: Typical domain structures resulted from several local perturbations for a 100×100 array. Each cluster is triggered by a single perturbation.

In a sense, we are measuring the linear response of the system under infinitesimal perturbations. The quantity being measured is the distribution function $D(s)$ of slide sizes. It revealed that

$$D(s) \approx s^{-\tau}, \ \tau \approx 0.98 \approx 1 \ for \ D = 2 \quad (8)$$

See Fig.3. At small sizes the curve deviates from the straight line because discreteness effects of the lattice come into play. While the fact that the distributions begin to deviate from a power law at large cluster sizes is a finite-size effect.

In order to understand the dynamics of the critical state, they also investigated the temporal evolution of the clusters above. Imagine first the effect of a perturbation at a single site on a static critical state. A local perturbation will spread to (some) nearest-neighbor sites, then to next-nearest neighbors, and so on in a "domino" effect, eventually dying out after a total time T, having induced a total of s slidings. In general, T is less than s since the growth rate is usually greater than unity. Fig.4 shows the distribution of lifetimes $D(T)$ weighted by the average response s/T.

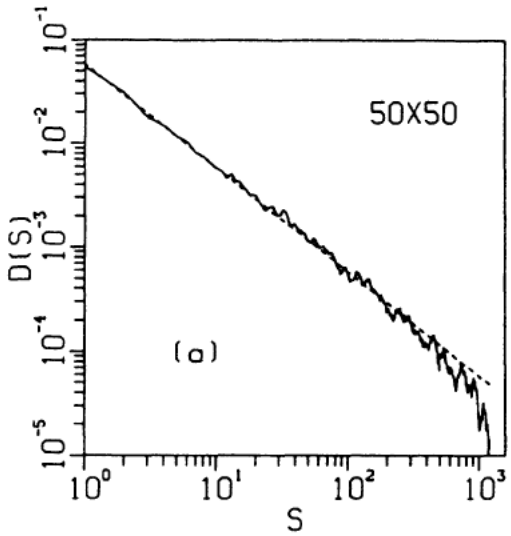

Figure 3: Distribution of cluster sizes at criticality in two dimensions.

This quantity also has power-law behavior,

$$D(T) \approx T^{-\alpha},\ \alpha \approx 0.43\ for\ D = 2. \tag{9}$$

We now give the "sliding" a new meaning, namely a point of energy dissipation: When a sliding event occurs, a unit of energy is dissipated. This will lead to $1/f$ noise for the power spectrum ($S(f) \approx f^{-2+\alpha}$) discussed before. Hence, the statement that the system must go to a critical state with power-law spatial correlations and $1/f$ noise is fully consistent with the numerical simulations. The $1/f$ noise is the temporal signature of the self-similar properties of the critical state.

Note that the curves for the lifetime distribution fit a power law only over a decade or so, while the cluster size distributions fit for at least two decades. This is due to the fact that the lifetime of a cluster is much smaller than its size, thus limiting the range over which we have reliable data in Fig.4. What's more, the exponents τ and α representing the spatial and temporal evolution of the clusters, respectively, can be related through "scaling relations"

$$\alpha = 2 - \beta = (\gamma + 1)\tau - 2\gamma \tag{10}$$

where $\gamma \approx 0.57$ for $D = 2$ and $\gamma \approx 0.71$ for $D = 3$ if the perturbation grows with an exponent γ within the clusters, i.e., $s \approx T^{1+\gamma}$.

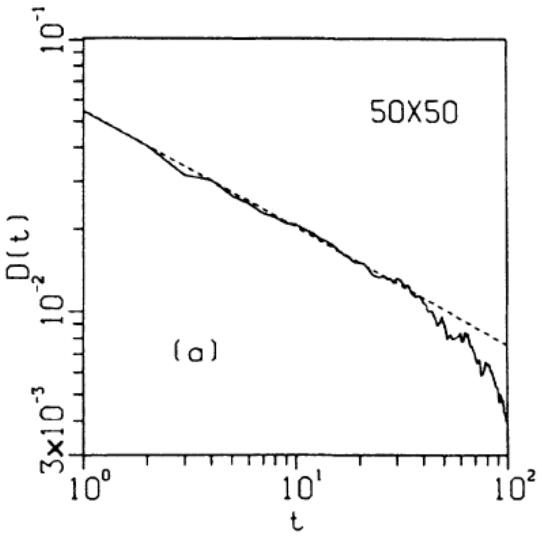

Figure 4: Distribution of lifetimes corresponding to Fig.3

3.2.3 How can SOC explain the $1/f$ noise and fractals?

The speculation by BTW was as follows. A signal will be able to evolve through the system as long as it is able to find a connected path of above threshold regions. When the system is either driven at random or started out from a random initial state, regions that are able to transmit a signal will form some sort of random network. This network will be modified, or correlated, by the action of the internal dynamics induced by the external drive. The dynamics stop every time the internal dynamics have relaxed the system, so that all local regions are below threshold. The slow external drive will eventually bring some region above threshold once again, and the internal relaxation will restart. The result is a complicated, delicately interwoven web of regions that are coupled dynamically. When we continue to drive the system after this marginally stable SOC state has been reached, we will see flashes of action as the external perturbation manages to spark off activity through different routes of the system. The intricate nature of the combined operation of the external drive and the internal relaxation of the threshold dynamics makes it natural to imagine that the network of connected dynamical paths has some sparse percolation-like geometry. It could well be that the structure of this dynamical network has a fractal geometry; at least this was the suggestion of BTW. If the activated regions consist of fractals of various sizes, then the duration of the induced relaxation processes traveling through these fractals can also be expected to vary greatly. It is well known that many different-acting time scales can, under certain circumstances, leads to $1/f$ noise. BTW imagined that this is precisely what happens in SOC systems.

3.3 Conservation laws and hydrodynamics model

As we can see, Eq.(7) is a nonlinear discretized diffusion equation (nonlinear because of the threshold condition). The question that which possible principles governing the behavior of SOC and how to relate it to our familiar dynamic model then arises. The first thing comes to our mind is, of course, the conservation laws since the particle number is conserved. The conservation laws, related to symmetries, is deemed as the origin of self-similarity of the diffusing field [7]. These bring us to the hydrodynamic theory/model of SOC, which is a continuum model and valid for large scales, long time scales.

We note that the important constraint that *the relaxation dynamics during an avalanche does not change the number of particles*, while the driving operation violates this conservation by adding particles randomly from the outside. Based solely on the above condition, we conclude that the equation of motion must take the form

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{j}(h) = \eta(\mathbf{x}, t).$$

The left-hand side of the equation represents the conservative (and deterministic) relaxation that follows the addition of particles, while the right-hand side represents the external sources and sinks in terms of a random input function η. As we said before, the slope can be called "pressure" for generality. So we can rewrite the conservation law for 1D case as

$$\frac{\partial \delta P}{\partial t} + \frac{\partial \Gamma(\delta P)}{\partial x} - D_0 \partial_x^2 P = \tilde{s} \tag{11}$$

where δP is the deviation of the pressure from the self-organized critical state, $\tilde{s}$ is the input noise and $\Gamma(\delta P)$ is flux induced by δP. The last term on the left hand side is coarse grains induced diffusion. Obviously, P is conserved so that δP evolves via $\nabla \cdot \Gamma$ only. To obtain Γ, we need to examine the underlying symmetries of the problem. Note that, with respect to the average flat surface, "bumps" move downhill while "voids" move uphill as illustrated in Fig.5. We therefore have the

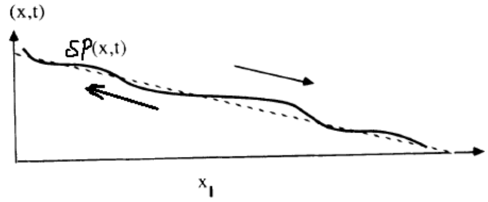

Figure 5: δP is defined as a deviation from critical state

joint reflection symmetry $\delta P \to -\delta P$ and $x \to -x$ remaining Γ unchanged. Then we can assume $\Gamma(\Delta P)$ has the form

$$\Gamma(\delta P) = \sum_{n,m,p,q,r} A_n(\delta P)^n + B_m(\frac{\partial \delta P}{\partial x})^m + C_p(\frac{\partial^2 \delta P}{\partial x^2})^p + D_{q,r}\delta P^q(\frac{\partial \delta P}{\partial x})^r + \cdots$$

The joint reflection symmetry ensure that n, p, q being even.

We are interested in the large-distance properties of the system, i.e., long wavelength (hydrodynamic) limit $k \to 0$. We expect the fluctuations of δP to be small if the surface is flat as initially assumed. Therefore higher-order terms in δP and spatial derivative are ignored, yielding the smoothest contribution:

$$\Gamma(\delta P) = A\delta P^2 - B\frac{\partial \delta P}{\partial x} \tag{12}$$

Here the negative sign before B is for convenience as diffusion equation. Here the first term is the driving force; this term originates from the local transport dynamics such as the nonlinear friction or the threshold dynamics. While the second term is the linear current present in any diffusive process; B can be interpreted as the surface tension for the sandpile for example. Thus the equation of motion becomes

$$\frac{\partial \delta P}{\partial t} + \frac{\partial}{\partial x}\left[\alpha \delta P^2 - D\frac{\partial \delta P}{\partial x}\right] = \tilde{s} \tag{13}$$

Here α and D are just coefficients and $D = D_0 + B$. It's like the classic Burger's equation. For 2D case, we have similar result

$$\frac{\partial \delta P}{\partial t} + \partial_{\|}\alpha\delta P^2 - D\partial_{\|}\delta P - \gamma\partial_{\perp}\delta P = \tilde{s} \tag{14}$$

where $\|$ and $\perp$ denote the direction that along slope and perpendicular slope, respectively. The only difference is the extra damping term. So we will examine the dynamics using 1D equation.

Before we present a detailed analysis of Eq.(13), we emphasize that its most important feature is the absence of a relaxation term of the form $-\delta P/\tau$. Such a term introduces a characteristic time τ, and a corresponding length $l=(\tau^2/D)^{1/2}$, and destroys scale invariance. It is the conservative nature of the deterministic dynamics that rules out this term in Eq.(13). What's more, the scale of δP is mesoscale. The last but not least, $\partial_x \alpha \delta P^2 \sim \delta P \partial_x \delta P$ indicates that the avalanche with big driving force "moves" fast than the small one, i.e., the avalanches overlap, and the resulting output signals can exhibit 1/f noise. Also, due to their overlap, in this regime it is no longer possible to identify single avalanches.

We can use Eq.(13) to calculate the frequency spectrum, re-normalized nonlinear scrambling of coupling time and lifetime of each mode/relaxation time, etc. For this, let's take its Fourier transform

$$-i\omega\delta P_{k,\omega}+N_{k,\omega}+Dk^2\delta P_{k,\omega}=\tilde{s}_{k,\omega} \tag{15}$$

where $N_{k,\omega}=ik\alpha\sum_{k',\omega'}\delta P_{-k',-\omega'}\delta P_{k+k',\omega+\omega'}$. Using the "beat term" idea from quasilinear theory, let

$$\delta P_{-k',-\omega'}\delta P_{k+k',\omega+\omega'}=\delta P_{-k',-\omega'}\delta P^{(2)}_{k+k',\omega+\omega'}$$

where the beat term $\delta P^{(2)}_{k+k',\omega+\omega'}$ satisfies

$$\left[-i(\omega+\omega')+(k+k')^2D_0+(k+k')^2\gamma_T\right]\delta P^{(2)}_{k+k',\omega+\omega'}=-i\alpha(k+k')\delta P_{k',\omega'}\delta P_{k,\omega}$$

Then we can reach

$$\begin{aligned}N_{k,\omega}&=ik\alpha\sum_{k',\omega'}\delta P_{-k',-\omega'}\frac{-i(k+k')\alpha\delta P_{k',\omega'}\delta P_{k,\omega}}{-i(\omega+\omega')+(k+k')^2\gamma_T}\\&\equiv k^2\gamma_T\delta P_{k,\omega}\end{aligned}$$

where we neglected D_0 relative to γ_T. The long wavelength and slow perturbation limit indicate that $k,\omega\to 0$, which gives

$$\gamma_T=\alpha^2\sum_{k',\omega'}|\delta P_{k',\omega'}|^2\frac{k'^2\gamma_T}{\omega'^2+(k'^2\gamma_T)^2} \tag{16}$$

γ_T itself sets lifetime of interaction. Plugging it into Eq.(15) yields the final expression

$$\gamma_T=\alpha^2\sum_{k',\omega'}\frac{|\tilde{s}_{k',\omega'}|^2}{(k'^2\gamma_T)^3}\frac{1}{[1+(\omega'/k'^2\gamma_T)^2]^2}$$

Now we can replace the sum by integral and assume $\tilde{s}$ is white noise, i.e., $|\tilde{s}_{k',\omega'}|^2 = s_0^2$ which generates

$$\gamma_T = \left[c_1 \alpha^2 s_0^2/3\right]^{1/3} k_{min}^{-1} \tag{17}$$

where c_1 is an integral constant. We can see that γ_T depends explicitly on the cut-off scale and grows with the increase of scale of interest $\gamma_T \sim \gamma_{T_0} \delta l$. And $(\delta x^2) \sim \gamma_T t$, $\gamma_T \sim k_{min}^{-1}$, thus $\delta x^2 \sim t k_{min}^{-1} \sim \delta x\, t$, $\delta x \sim t$, which indicates that δP pulse propagates ballistically rather than diffusively.

4 Schelling Segregation Model

In 1969, Thomas C. Schelling developed a simple but striking model of racial segregation [8]. His model studies the dynamics of racially mixed neighborhoods, showing how local interactions can lead to surprising aggregate structure. In particular, it shows that relatively mild preference for neighbors of similar race can lead in aggregate to the collapse of mixed neighborhoods, and high levels of segregation.

It is about the segregation that can result from discriminatory individual behavior. It examines some of the individual incentives, and perceptions of difference, that can lead collectively to segregation. The basic model is as follows: Suppose there is some area that both blacks and whites would prefer to occupy as long as the ratio of opposite color to one's own color does not exceed some limit-tolerance; Assume that anyone whose limiting ratio is exceeded by the prevailing mixture will go elsewhere following some rules, for example, a dissatisfied member moves to the nearest point that meets his minimum demand. Nobody in this model anticipates the movements of others. So we also need an order of moving; arbitrarily let the discontented members move in turn. The result is segregation forms. The initial distribution of the two populations and the rates at which they move in or out will determine which one of the two colors eventually occupies and which one evacuates.

For example, suppose there are two types of agents: X and O. The two types of agents might represent different races, ethnicity, economic status, etc. Two populations of the two agent types are initially placed into random locations of a neighborhood represented by a grid. After placing all the agents in the grid, each cell is either occupied by an agent or is empty as Fig.6. The threshold (tolerance) t is one that will apply to all agents in the model, even though in reality everyone might have a different threshold they are satisfied with. Note that the higher the threshold, the higher the likelihood the agents will not be satisfied with their current location. Let's assume a threshold t of 30%. This means every agent is fine with being in the minority as long as there are at least 30% of similar agents in adjacent cells. The picture in Fig.6 (left) shows a satisfied agent because 50% of X's neighbors are also X (50% $> t$). The next X (right) is not satisfied because only 25% of its neighbors are X (25% $< t$). Notice that in this example empty cells are not counted when calculating similarity. When an agent is not satisfied, it can be moved to any vacant location in the grid. Any algorithm can be used to choose this new location. For example, a randomly selected cell may be chosen, or the agent could move to the nearest available location. The new configuration may cause some agents which were previously satisfied to become dissatisfied! All dissatisfied agents must be moved in the same *round*. After the round is complete, a new round begins, and dissatisfied agents are once again moved to new locations in the grid. These rounds continue until all agents in the neighborhood are satisfied with their location and clusters form.

You can experiment with a number of parameters and see how the model behaves in the following website Schelling's Model of Segregation.

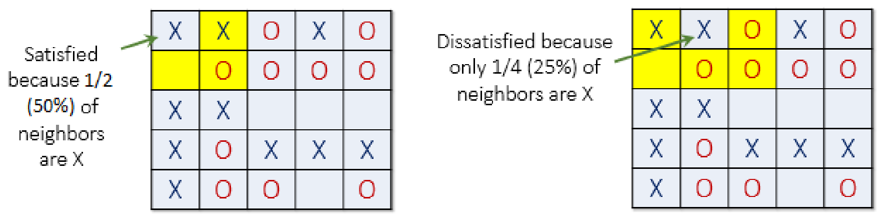

Figure 6: Occupation of an area by two agent types whose threshold is 30%.

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