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Path integrals and quantum interference

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The path integral treatment of the single and double-slit experiments, which illustrates the particle nature of single events and the wave nature of the statistical average of repeated events, is presented.

I. INTRODUCTION

Interference phenomenon is a simple but crucial *universal* behavior of light and other particles involving either a beam of particles, or a large number of repeated events collected at a screen. It is sufficient to consider the celebrated prototypes, namely the two-slit interference and one-slit diffraction experiments. There are of course, innumerable discussions of these experiments. And the more different ways one can describe the phenomena, the better will be our understanding. The path integral approach[1–4] provides yet another useful insight and we are not aware of a complete discussion of one- and two-slits experiments via path integrals, although the method has now been applied to a very large number of problems.[5]

The present work originated from recent discussions on the impossibility of assigning definite values of observables to individual events[6–9] and the need of distinguishing clearly between a theory of single events and a theory of repeated events[10] which alleviates the quantum paradoxes. In the path integral formalism, we deal on the one hand with individual trajectories but then form an average to obtain the final quantum result. However, we shall emphasize that these trajectories are not all "physical" and discuss the relation of these to the Huygens' principle of wave propagation. We also indicate the formulation of the notion of "coherence" in the path integral formalism.

II. PATH INTEGRALS: SINGLE AND REPEATED EVENTS

According to the path integral formalism, every quantum system is described as a particle, and the probability that it goes from position x_1 at time t_1, to a position x_2 at time t_2 is given by

$$P(2,1)=|K(2,1)|^2, \tag{1}$$

where the probability amplitude (propagator) $K(2,1)$ is calculated from

$$K(2,1)=N_{21}\sum_a \exp\left(\frac{1}{\hbar}S_a(t_2,t_1)\right), \tag{2}$$

where N_{21} is a normalization constant,

$$S_a(t_2,t_1)=\int_{t_1}^{t_2} L[t,x_a(t),\dot{x}_a(t)]\,dt, \tag{3}$$

is the classical action for path a,L the classical Lagrangian, and the sum is over all paths a from $x_1=(\mathbf{x}_1,t_1)$ to $x_2=(\mathbf{x}_2,t_2)$. For relativistic problems, we may use instead of t an invariant parameter τ and take x to be the Minkowski four vector. All formulas remain essentially the same.

Among the paths with endpoints fixed in space and time, there is only one for which the classical action S_{cl} is an extremum, and this is the unique classical trajectory. Classical mechanics corresponds to the dominance of the contribution of this path to the sum in Eq. (2) when $S_c \gg \hbar$ (Ref. 11).

The summation over all paths in Eq. (2) and the square of the sum in Eq. (1) are the sources of the interference terms that make the behavior of repeated events different than in classical mechanics.

The different paths (alternatives) in Eq. (2) are not assigned definite probabilities but complex-valued amplitudes that are additive and lead, when squared, to the observed probabilities in repeated experiments. However, one can still continue to think of the behavior of a single particle, but at the expense of a probabilistic language. This does not necessarily imply that a description of a single event (when not repeated) is not possible.[12] We shall continue to use in the following the standard notion of "particle" in quantum theory. Feynman states[13] "I like to emphasize that light comes in this form-particles. It is very important to know that light behaves like particles, especially for those of you who have gone to school, where you were probably told something about light behaving like waves. I'm telling you the way it *does* behave—like particles." The same statement applies to electrons, nucleons, and all other elementary quantum systems. This is confirmed experimentally by the fact that anytime a *single* quantum particle is observed it is found to be highly localized, and that wave properties are strictly associated with the statistical behavior of particles. There is no way of determining the wavelength of a particle characterizing an interference pattern by the observation of a single event.

Any experimental determination of which alternative a particle follows among several possible alternatives, will destroy the interference between the alternatives. This fact can be taken to be the essence of the uncertainty principle.[14]

If a path a from x_1 to x_2 consists of two successive segments with an intermediate point x_3, then it follows from the definition (3) that

$$S_a(t_2,t_1)=S_a(t_2,t_3)+S_a(t_3,t_1). \tag{4}$$

This and (2) then imply[15] that

$$K(2,1)=\int K(2,3)K(3,1)\,dx_3\,. \tag{5}$$

For a free nonrelativistic particle of mass m, the propagator (2) is calculated to be, as is well known,[16]

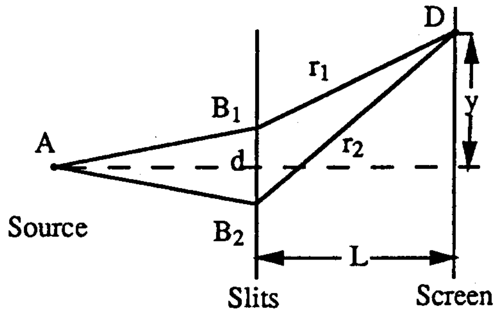

Fig. 1. Two paths through double slits at B_1 and B_2.

$$K(2,1)=\left(\frac{m}{ih(t_2-t_1)}\right)^{3/2}\exp\left(\frac{im}{2\hbar}\frac{(x_2-x_1)^2}{t_2-t_1}\right)=N_{21}e^{i\Phi}, \tag{6}$$

which is the Green's function of the Schrödinger equation also calculable by other methods. Similarly the propagators corresponding to other wave equations can be evaluated.[17]

III. THE TWO-SLIT EXPERIMENT

Consider a source (A) of particles of momentum p, which can be photons, electrons, nucleon, etc. (Fig. 1). Let A be symmetrically placed behind the two slits (B_1,B_2) and the particles be detected by microdetectors (D) placed on the other side of the slits. Then, according to Eq. (5) and choosing an intermediate time at the slits corresponding to the assumption of the dominance of the classical trajectory,

$$K(D,A)=K(D,B_1)K(B_1,A)+K(D,B_2)K(B_2,A). \tag{7}$$

By symmetry, Δt and Δx are the same for paths B_1A and B_2A, and hence

$$K(B_1,A)=K(B_2,A)\equiv K_0. \tag{8}$$

For a free particle, the normalization constant N in Eq. (2) is a function of the mass of the particle and the time interval (t_2-t_1) [see Eq. (6)]. Thus for the two paths from B_1 and B_2 to D, we have

$$N_1=N_2\equiv N,$$

because the total time interval in $K(D,A)$ is fixed and $t(A,B_1)=t(A,B_2)$ in Eq. (8). Consequently,

$$\begin{aligned}K(D,A)&=NK_0\left[\sum_a\exp\left(\frac{i}{\hbar}S_{1_a}\right)+\sum_a\exp\left(\frac{i}{\hbar}S_{2_a}\right)\right]\\&=NK_0(e^{i\Phi_1}+e^{i\Phi_2}).\end{aligned} \tag{9}$$

If we let

$$|NK_0|\equiv I^{1/2}, \tag{10}$$

then according to Eq. (1),

$$P(D,A)=|K(D,A)|^2=2I(1+\cos\theta), \tag{11}$$

where

$$\theta\equiv|\Phi_1-\Phi_2|. \tag{12}$$

If the interference term $\cos\theta$ were zero, then $P(D,A)=2I$, and thus I can be interpreted to be the intensity from each individual slit. The maxima and minima of the total intensity are

$$P(D,A)=4I \quad \text{at} \quad \cos\theta=1, \quad \theta=0,2\pi,4\pi,..., \tag{13}$$

and

$$P(D,A)=0 \quad \text{at} \quad \cos\theta=-1, \quad \theta=\pi,3\pi,5\pi,... .$$

According to Eqs. (6) and (12),

$$\theta=\frac{m}{2\hbar\tau}(r_1^2-r_2^2)=\frac{m}{\hbar}\frac{r_1+r_2}{2\tau}(r_1-r_2)=\frac{p}{\hbar}\Delta r, \tag{14}$$

where $r\equiv(1/2)(r_1+r_2)$ is the mean distance, $u=r/\tau$ is the speed, and $p=mu$ is the momentum. For the first maximum, $\theta=2\pi$, $\Delta r\equiv r_1-r_2=\lambda$, and we obtain from Eqs. (14) and (13),

$$\lambda=h/p. \tag{15}$$

The assumption that τ is the same for both paths, although the path lengths are different, is the path integral formulation of complete coherence. This is true only for stationary monochromatic beams, and agrees with Huygens' principle for stationary propagation, and stationary scattering theory which in turn means that the events are repeated uniformly.

What is truly remarkable is that Eq. (15) holds experimentally for *any* type of incident particle of momentum p, whether it is a photon, electron, nucleon, etc., in spite of the fact that the interactions of the different particles with the slit system may be very different.

Theoretically, the derivation of Eq. (15) by means of path integrals for photons, relativistic massive particles, and particles with nonzero spin is complicated.[18] However, in the Schrödinger picture of quantum mechanics, Eq. (15) is usually derived as follows: The state of the particles leaving each slit is essentially that of a momentum eigenstate modulated by the diffraction effect of the slit.

$$\begin{aligned}&\psi_j(\mathbf{r}_j,t)=f_j(\mathbf{r}_j)\exp[i(\omega t-\mathbf{k}_j\cdot\mathbf{r}_j)],\\&\mathbf{r}_1=\mathbf{r}-\mathbf{d}/2, \quad \mathbf{r}_2=\mathbf{r}+\mathbf{d}/2,\end{aligned} \tag{16}$$

where

$$\mathbf{k}_j=k\mathbf{r}_j/r_j, \quad k=p/\hbar=2\pi/\lambda. \tag{17}$$

The wavefunction at the screen is then, for stationary states,

$$\psi=\psi_1+\psi_2=e^{i\omega t}(f_1e^{-ikr_1}+f_2e^{-ikr_2}), \tag{18}$$

which is again the Huygens' principle. The intensity is given by

$$|\psi|^2=|f_1|^2+|f_2|^2+2|f_1f_2|\cos[k(r_1-r_2)]. \tag{19}$$

For $d\ll L$, $f_1\simeq f_2\simeq f$, and

$$|\psi|^2=2|f|^2(1+\cos\theta), \tag{20}$$

where

$$\theta\equiv k(r_1-r_2)=2\pi(r_1-r_2)/\lambda. \tag{21}$$

The first maximum occurs at $\theta=2\pi$, or $(r_1-r_2)=\lambda=h/p$, in agreement with Eq. (15).

As the intensity of the source gets weaker and weaker, the intensity (20) at the screen does *not* maintain its spatial

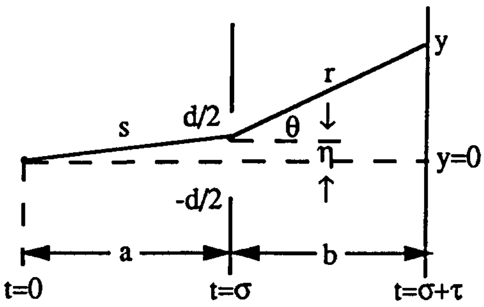

Fig. 2. A typical path in a single slit diffraction.

distribution with weaker intensity. Instead, a point is reached where single localized pulses are detected on the screen, indicating that only one particle at a time is arriving. The statistical distribution of the pulses accumulated over a period of time is what is described by Eq. (20). In this situation, if detectors are placed immediately behind the slits, it is found that when one detector is triggered, the other is not; implying that a particle is not some kind of wave that overlaps both slits. Moreover, if it is determined experimentally through which slit a particle goes through, then the intensity at the screen is not given by Eq. (20), but rather by

$$|\psi|^2=|\psi_1|^2+|\psi_2^2|=2|f|^2$$

(i.e., by a superposition of the *intensities* of the individual slits, and the interference effect is destroyed). All this illustrates the remarks made at the end of Sec. 1.

IV. DIFFRACTION BY A SINGLE SLIT

Consider the diffraction of a nonrelativistic particle of mass m by a slit of width d (Fig. 2). Again, all trajectories are assumed to have the same time intervals τ and σ although the path lengths are different. According to Eqs. (5) and (6) and again under the property of the dominance of the classical trajectory corresponding to intermediate time at the slit,

$$\begin{aligned}K(y,\sigma+\tau;0,0) &= \int K(y,\sigma+\tau;\eta,\sigma)K(\eta,\sigma;0,0)d\eta\\ &\equiv K(y)\\ &= N_a N_b \int_{-d/2}^{d/2} \exp\left(\frac{imr^2}{2\hbar\tau}\right)\exp\left(\frac{ims^2}{2\hbar\sigma}\right)d\eta,\end{aligned} \tag{22}$$

where

$$s^2=a^2+\eta^2,\quad r^2=b^2+(y-\eta)^2.$$

Thus

$$\frac{m}{2\hbar}\left(\frac{s^2}{\sigma}+\frac{r^2}{\tau}\right)=A(\eta-\eta_0)^2+B,$$

where

$$A=\frac{m}{2\hbar}\frac{\sigma+\tau}{\sigma\tau},\quad \eta_0=\frac{\sigma\tau}{\sigma+\tau}\frac{y}{\tau}, \tag{23}$$

and

$$B\equiv\frac{m}{2\hbar}\left(\frac{a^2}{\sigma}+\frac{b^2+y^2}{\tau}-\frac{\sigma}{\sigma+\tau}\frac{y^2}{\tau}\right).$$

Substituting Eq. (23) into Eq. (22), we obtain

$$K(y)=N_A N_b e^{iB}\int_{-d/2}^{+d/2} e^{iA(\eta-\eta_0)^2}d\eta. \tag{24}$$

Let

$$A(\eta-\eta_0)^2\equiv\frac{\pi}{2}iZ^2,\quad Z_\pm\equiv\left(\frac{2A}{\pi}\right)^{1/2}\left(\frac{d}{2}\pm\eta_0\right), \tag{25}$$

$$E(\zeta)\equiv\int_0^\zeta \exp\left(i\frac{\pi}{2}Z^2\right)dZ=C(\zeta)+iS(\zeta), \tag{26}$$

$$C(\zeta)\equiv\int_0^\zeta \cos\left(\frac{\pi}{2}Z^2\right)dZ,$$

$$S(\zeta)\equiv\int_0^\zeta \sin\left(\frac{\pi}{2}Z^2\right)dZ, \tag{27}$$

where the integrals in Eq. (27) are the Fresnel integrals. Then,

$$K(y)=N_a N_b e^{iB}\left(\frac{\pi}{2A}\right)[E(Z_-)+E(Z_+)], \tag{28}$$

and the Fresnel diffraction intensity is given by $|K(y)|^2$.

For Fraunhofer diffraction,

$$s\to\infty,\quad \sigma\to\infty,\quad A\to(m/2\hbar\tau),\quad \eta_0\to y. \tag{29}$$

If we assume that

$$\eta<d/2\ll y, \tag{30}$$

then

$$A(\eta-\eta_0)^2\to\frac{m}{2\hbar\tau}(\eta-y)^2\simeq\frac{m}{2\hbar\tau}(y^2-2y\eta), \tag{31}$$

and Eq. (24) gives

$$K(y)=I^{1/2}e^{iC}(\sin\alpha)/\alpha, \tag{32}$$

where (see Fig. 2), for $d\ll y$,

$$\alpha\equiv\frac{m}{\hbar}\frac{y}{\tau}\frac{d}{2}\simeq\pi d\frac{m}{h}\frac{r}{\tau}\sin\theta=\pi d\frac{p}{h}\sin\theta=\frac{\pi d}{\lambda}\sin\theta, \tag{33}$$

$$I^{1/2}\equiv N_a N_b d,\quad C\equiv\frac{m}{2\hbar}\left(\frac{a^2}{\sigma}+\frac{b^2+y^2}{\tau^2}\right)\simeq\frac{m}{2\hbar}\frac{b^2+y^2}{\tau^2}. \tag{35}$$

The intensity $|K(y)|^2$ agrees with the usual expression. Note that $u=r/\tau$ is the speed, $p=mu=mr/\tau$ is the momentum, and $\lambda=h/p$ is the wavelength.

This completes our discussion of the single and double-slit interference phenomena from the point of view of path integration.

V. SUMMARY

In the Schrödinger method one obtains the quintessential quantum phenomena of interference and diffraction as a purely wave properties, i.e., solution of the wave equation for the given boundary conditions. In the path integral method, a complex probability amplitude is assigned to each path between the source and detector and then these amplitudes are summed. This procedure is reminiscent but different from considering single events on the screen and then taking the average over repeated events. The present work, we hope, clarifies some of the points related to quantum interference, path integrals and single versus repeated events relationships.

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[1]R. P. Feynman, and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[2]R. P. Feynman, *QED: The Strange Theory of Light and Matter* (Princeton U.P., New Jersey, 1985).
[3]L. S. Schulman, *Techniques and Applications of Path Integrals* (Wiley, New York, 1981).
[4]L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Benjamin, New York, 1980).
[5]Third Int'l Conference on *Path Integrals from meV to MeV*, edited by V. Sa-yakanit, W. Sritrakool, J.-O. Berananda, M. C. Gutzwiller, A. Inomata, S. Lundquist, J. R. Klauder, and L. Schulman (World Scientific, Singapore, 1989).
[6]S. Kochen and E. P. Specker, "The problem of hidden variables in quantum mechanics" J. Math. Mech. **17**, 59–88 (1967).
[7]N. D. Mermin, "Simple unified form for the major no-hidden-variables theorem," Phys. Rev. Lett. **65**, 3373–3376 (1990).
[8]A. Peres, "Incompatible results of quantum measurements," Phys. Lett. A **151**, 107–108 (1990).
[9]D. M. Greenberger, M. Horne, A. Shimony, and A. Zeilinger, "Bell's theorem without inequalities," Am. J. Phys. **58**, 1131–1143 (1990).
[10]A. O. Barut, New York Acad. Sci. "Towards a theory of single events in spin correlation experiments," **480**, 393–399 (1987); "Quantum Theory of Single Events: de Broglie wavelets, Schrödinger waves and classical trajectories," Found. Phys. **20**, 1233–1240 (1990).
[11]See Ref. 1, pp. 29–31; Ref. 2. pp. 53–56.
[12]A. O. Barut, *Symposium on the Foundations of Modern Physics*, edited by P. Lahti, P. Mittelstaedt (World Scientific Singapore, 1991), p. 31.
[13]See Ref. 2, p. 15.
[14]Ref. 1, p. 9, Ref. 2, pp. 55–56.
[15]Ref. 1, p. 36.
[16]Ref. 1, pp. 42–43, 78.
[17]Ref. 3, pp. 164, 225.
[18]Ref. 3, Chaps. 20, 22, 25.

Solutions of Maxwell's equations for electric and magnetic fields in arbitrary media

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Solutions of Maxwell's equations for fields in arbitrary media are derived in terms of charge density ρ, current density **J**, polarization **P**, and magnetization **M**. The solutions express **E**, **D**, **H**, and **B** as integrals of retarded ρ, **J**, **P**, **M**, and their spatial and temporal derivatives.

I. INTRODUCTION

Most of the previously reported solutions of Maxwell's equations as integrals of retarded charge and current densities were limited, according to their authors, to a vacuum or to media of constant permittivity and permeability occupying all space.[1–3] Griffiths and Heald, who discussed such solutions and provided their own derivations,[4] stated in their footnote 22 that the solutions could be extended to fields in dielectric and magnetic media, if "one interprets **J** to include $\partial \mathbf{P}/\partial t$ and $\nabla\times\mathbf{M}$ in addition to the free current density $\mathbf{J}_{\text{free}}$ and ρ to include $-\nabla\cdot\mathbf{P}$ in addition to the free charge density ρ_{free}." However, they did not actually provide or discuss the extended solutions.

The purpose of this paper is to present detailed derivations of the solutions of Maxwell's equations for fields in arbitrary media and to demonstrate their possible applications. Two sets of solutions are obtained. The first set expresses the fields **E**, **D**, **H**, and **B** in terms of retarded charge density ρ, retarded current density **J**, retarded polarization **P**, retarded magnetization **M** and retarded spatial and temporal derivatives of ρ, **J**, **P**, and **M**. The second set contains no spatial derivatives.

The solutions are general and impose no restrictions on any of the quantities involved except that **E**, **D**, **H**, and **B** are assumed to be regular at infinity and that ρ, **J**, **P**, and **M** are assumed to be confined to a finite region of space. Since these requirements are implicit in Maxwell's equations, the solutions are equivalent to Maxwell's equations.

II. BASIC EQUATIONS AND DEFINITIONS

The equations to be solved are the four Maxwell's equations:

$$\nabla\cdot\mathbf{D}=\rho, \tag{1}$$

$$\nabla\cdot\mathbf{B}=0, \tag{2}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}. \tag{4}$$

In order to solve these equations we also need equations correlating **D** with **E** and **B** with **H**. The most general equations of this type are those making use of the polarization **P** and magnetization **M**. They are

$$\mathbf{P}=\mathbf{D}-\epsilon_0\mathbf{E}, \tag{5}$$

$$\mathbf{M}=\mathbf{B}-\mu_0\mathbf{H}. \tag{6}$$

[If **P** and **M** are independently defined (as dipole moment densities, for example), then these equations constitute def-