Electromagnetic waves in conductors

Ohm's law $\vec{J}_f = \sigma \vec{E}$ (o)

$\Rightarrow$ M.E.: $\nabla \cdot \epsilon \vec{E} = \rho_f$ (i) $\quad \nabla \times \vec{E} = -\partial_t \vec{B}$ (iii)

$\nabla \cdot \vec{B} = 0$ (ii) $\quad \nabla \times \frac{1}{\mu}\vec{B} = \sigma \vec{E} + \epsilon \partial_t \vec{E}$ (iv) $\quad \tau \equiv \epsilon/\sigma$

Continuity eqn. for free charge $\nabla \cdot \vec{J}_f = -\partial_t \rho_f$

(o) $\Rightarrow \partial_t \rho_f = -\sigma\, \nabla \cdot \vec{E} = -\frac{\sigma}{\epsilon}\rho_f \xrightarrow{\text{homog. medium}} \rho_f(t) = e^{-\sigma/\epsilon\, t}\rho_f(0)$

initial free charge dissipates

τ : how good a conductor is "perfect" $\sigma = \infty$, $\tau = 0$

good $\tau \ll$ other times scales $\approx \frac{1}{\omega}$ e.g.

"poor" $\tau \gg 1/\omega$

this is transient behavior just go $\rho_f = 0$

$\Rightarrow$ M.E.: $\nabla \cdot \epsilon \vec{E} = 0$ (i) $\quad \nabla \times \vec{E} = -\partial_t \vec{B}$ (iii)

$\nabla \cdot \vec{B} = 0$ (ii) $\quad \nabla \times \frac{1}{\mu}\vec{B} = \epsilon \partial_t \vec{E} + \underbrace{\sigma \vec{E}}_{\text{new term}}$ (iv)

new term w.r.t. linear media nonconductive.

$\nabla \times$ (iii) & (iv) $\Rightarrow \nabla^2 \vec{E} = \mu\epsilon \partial_t^2 \vec{E} + \mu\sigma \partial_t \vec{E}$

$\nabla \times$ (iv) & (iii) $\Rightarrow \nabla^2 \vec{B} = \mu\epsilon \partial_t^2 \vec{B} + \mu\sigma \partial_t \vec{B}$

planewave solutions: $\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z - \omega t)}$

$\tilde{B} \qquad \tilde{B}_0$

$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \qquad \boxed{\tilde{k}} = k + i\underline{\underline{K}} \Rightarrow k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{1/2}$

attn. $\quad K = \qquad -1$

$\tilde{E}(z,t) = \tilde{E}_0 e^{-Kz} e^{i(kz-\omega t)}$

$\tilde{B}(z,t) = \tilde{B}_0 \quad .$

$\boxed{\text{perfect conductor: } \sigma = \infty \Rightarrow k = \infty}$

skin depth $d \equiv \frac{1}{K}$

how far wave penetrates into conductor.

k determines wavelength, propagation speed & n index of refr.

$\lambda = \frac{2\pi}{k} \qquad v = \frac{\omega}{c} \qquad n = \frac{ck}{\omega} \left(= \frac{c}{v}\right)$

(i) & (ii) $\Rightarrow$ waves are transverse i.e. $\vec{E}$ polarized along x

$\tilde{E}(z,t) = \tilde{E}_0 e^{-Kz} e^{i(kz-\omega t)} \hat{x}$

(iii) $\Rightarrow \tilde{B} \qquad \tilde{B}_0 \qquad \hat{y}$

$\boxed{\tilde{k} = K e^{i\phi}} \quad K \equiv |\tilde{k}| = \sqrt{k^2 + K^2} = \omega\sqrt{\epsilon\mu\sqrt{1+(\sigma/\epsilon\omega)^2}} \qquad \phi \equiv \tan^{-1}(K/k)$ kappa/k

$\tilde{E}_0 = E_0 e^{i\delta_E} \qquad \delta_B - \delta_E = \phi$ magn. lags behind elect.

$\tilde{B}_0 = B_0 e^{i\delta_B} \qquad \frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}$

$\vec{E}(z,t) = E_0 e^{-kz} \cos(kz - \omega t + \delta_E)\, \hat{x}$

$\vec{B} \qquad = B_0 \qquad \delta_E + \phi)\, \hat{y}$

Reflection at a conducting surface

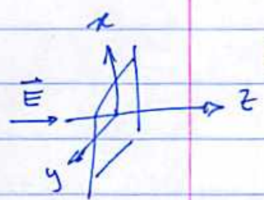
<u>Boundary conditions:</u>

free surface charge

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f \quad \text{(i)} \qquad\qquad \bar{E}_1^\parallel - \bar{E}_2^\parallel = 0 \quad \text{(iii)}$$

free surf. current

$$B_1^\perp - B_2^\perp = 0 \quad \text{(ii)} \qquad\qquad \frac{1}{\mu}\bar{B}_1^\parallel - \frac{1}{\mu_2}\bar{B}_2^\parallel = \bar{K}_f \times \hat{n} \quad \text{(iv)}$$



<u>Mono P.W.:</u>

$$\vec{E}_I(z,t) = \tilde{\vec{E}}_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{x}$$

$$\tilde{\vec{B}}_I \qquad \frac{1}{v_1} \qquad \hat{y}$$

$$\vec{E}_R(z,t) = \tilde{\vec{E}}_{0R}\, e^{i(-k_1 z - \omega t)}\, \hat{x}$$

$$\vec{B}_R(\qquad -\frac{1}{v_1} \qquad \hat{y}$$

$$\vec{E}_T(z,t) = \hat{\tilde{E}}_{0T}\, e^{i(\tilde{k}_2 z - \omega t)}\, \hat{x}$$

$$\bar{B}_T \qquad \frac{\tilde{k}^2}{\omega}\tilde{E}_{0T}\, e^{i(\tilde{k}_2 z - \omega t)}\, \hat{y}$$

} attn as it penetrates conductor.

@ $z=0$ $\quad E_\perp = 0$ both sides $\quad$ (i) $\Rightarrow$ $\sigma_f = 0$

$B_\perp = 0$ $\quad$ (ii) autom. satisf.

(iii) $\Rightarrow$ $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

(iv) $\bar{K}_f = 0 \Rightarrow \frac{1}{\mu_1 v_1}\left(\tilde{E}_{0I} - \tilde{E}_{0R}\right) - \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0T} = 0$

or $\quad \tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta}\tilde{E}_{0T} \qquad \tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$ $\quad$ compare to linear media interface $\frac{\mu_1 v_1}{\mu_2 v_2}$

Then

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I} \qquad \tilde{E}_{0T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0I} \qquad \text{similar to}$$

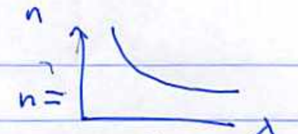
Perfect conductor $\quad \sigma = \infty \quad \tilde{k}_2 = \infty$

$$\tilde{E}_{0R} = -\tilde{E}_{0I} \qquad \& \qquad \tilde{E}_{0T} = 0$$

<u>Mirror</u>

e.g silver $\quad d \sim 100\,\text{Å}$

Frequency dependence of permittivity



Dispersion: e.g. prism. $n = \frac{ck}{\omega} \cong \sqrt{\epsilon_r}$ function of wavelength.

wave velocity $v = \frac{\omega}{k}$ group velocity $v_g = \frac{d\omega}{dk}$

Electrons in a nonconductor:

$$m\, d_t^2 x = F_{binding} + F_{damping} + F_{driving} \rightarrow e^{i\omega t}$$
$$= -m\omega_0^2 x - m\gamma\, d_t x + q E_0 \cos(\omega t) \qquad \tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$$
$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0$$

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$$
dipole moment. p out of phase with E by angle $\tan^{-1}\left[\gamma\omega/(\omega_0^2 - \omega^2)\right]$ small $\omega \ll \omega_0$, π $\omega \gg \omega_0$

molecule or atom & more transitions
$$\vec{\tilde{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\vec{\tilde{E}} \quad (N \text{: mol/volume})$$
$$= N\tilde{p}_0$$

Real part of: $\vec{\tilde{P}} = \epsilon_0 \tilde{\chi}_e \vec{\tilde{E}}$ $\quad \tilde{\epsilon} = \epsilon_0(1 + \tilde{\chi}_e)$ complex dielect. constant

$$\tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{Nq^2}{m\epsilon_0}\sum \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

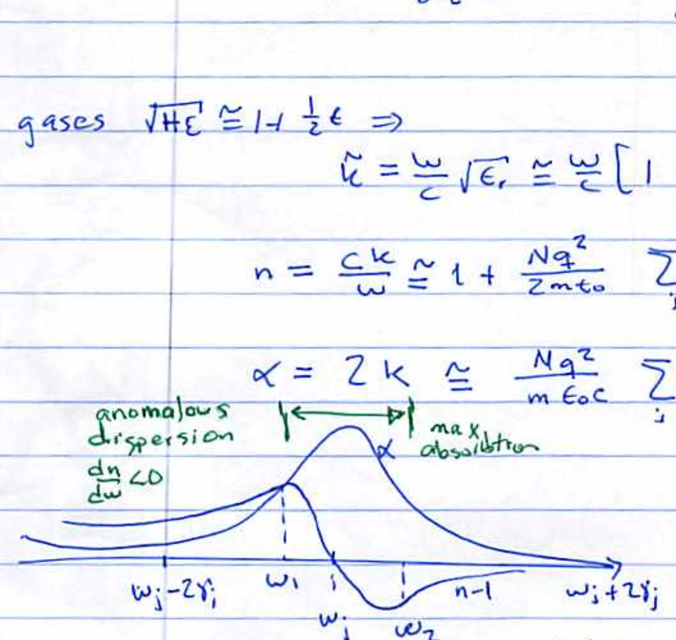
$\Rightarrow$ dispersive medium
$$\nabla^2 \vec{\tilde{E}} = \tilde{\epsilon}\mu_0 \frac{\partial^2 \vec{\tilde{E}}}{\partial t^2}$$
plane wave soln. $\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_0 e^{i(\tilde{k}z - \omega t)}$, $\tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\,\omega \equiv k + i\kappa$

absorption coeff. $\alpha \equiv 2\kappa$
intensity ~~prop~~ prop. to $E^2 \Rightarrow e^{-2\kappa z}$
here from damped oscill. NOT conductivity

gases $\sqrt{1+\epsilon} \cong 1 + \frac{1}{2}\epsilon \Rightarrow$
$$\tilde{k} = \frac{\omega}{c}\sqrt{\epsilon_r} \cong \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$
usually $\frac{dn}{d\omega} > 0$

$$\alpha = 2\kappa \cong \frac{Nq^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$



$|\rightarrow n < 1$? but there are more terms in the sum.

... other notes.