(11)

10.6 C

$$\tilde{\vec{p}} = \frac{Nq^2}{m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}\,\tilde{\vec{E}}$$

$$\tilde{p} = \epsilon_0 \tilde{\chi}_e \tilde{E}$$

complex susceptibility

Complex Dielectric Constant

$$\tilde{\epsilon}_r(\omega) = \frac{\tilde{\epsilon}(\omega)}{\epsilon_0} = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}$$

$$\nabla^2 \tilde{\vec{E}} = \tilde{\epsilon}\mu_0 \frac{\partial^2 \tilde{\vec{E}}}{\partial t^2}$$

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)}$$

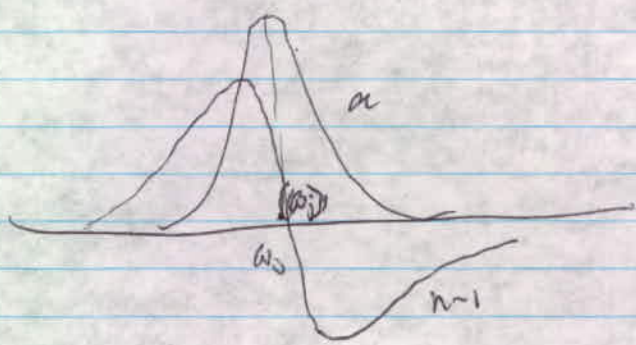
$$\tilde{k} = k + i\kappa \qquad n = \frac{ck}{\omega}$$

$$E(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

$$\tilde{k} = \frac{\omega}{c}\sqrt{\epsilon_r} \simeq \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{(\omega_j^2-\omega^2)-i\gamma_j\omega}\right.$$

$$n = \frac{ck}{\omega} \simeq 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}$$

$$\alpha = 2\kappa = \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}$$



Away from resonances

$$n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2}$$

For transparent materials, ω_j all lie in ultraviolet

so $\omega < \omega_j$

$$\frac{1}{\omega_j^2-\omega^2} = \frac{1}{\omega_j^2}\left(1-\frac{\omega^2}{\omega_j^2}\right)^{-1} \simeq \frac{1}{\omega_j^2}\left(1+\frac{\omega^2}{\omega_j^2}\right)$$

$$\therefore\ n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2} + \omega^2\left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^4}\right) \simeq 1 + A\left(1+\frac{B}{\lambda^2}\right)$$

Cauchy's Formula

Wave guides

Perfect conductor $\vec{E}=0$ $\vec{B}=0$ inside material of conductor

so Inner wall $\vec{E}^{\parallel}=0$

$\vec{B}^{\perp}=0$

$$\vec{E}(x,y,z,t)=\tilde{E}_0(x,y)\,e^{i(kz-\omega t)}$$
$$\tilde{B}(x,y,z,t)=\tilde{B}_0(x,y)\,e^{i(kz-\omega t)}$$

Interior of waveguide

$$\nabla\cdot\vec{E}=0 \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B}=0 \qquad \nabla\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\tilde{E}_0=E_x\hat{x}+E_y\hat{y}+E_z\hat{z}$$
$$\tilde{B}_0=B_x\hat{x}+B_y\hat{y}+B_z\hat{z}$$

Putting into Curl eqns

(i) $\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z$ (iv) $\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-\frac{i\omega}{c^2}E_z$

✓ (ii) $\frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x$ ✓ (v) $\frac{\partial B_z}{\partial y}-ikB_y=-\frac{i\omega}{c^2}E_x$

✓ (iii) $ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y$ ✓ (vi) $ikB_x-\frac{\partial B_z}{\partial x}=-\frac{i\omega}{c^2}E_y$

① $E_x=\frac{i}{(\frac{\omega}{c})^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right)$ (iii) + (v)

② $E_y=\frac{i}{(\frac{\omega}{c})^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right)$ (ii) + (vi)

③ $B_x=\frac{i}{(\frac{\omega}{c})^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$ (ii), (vi)

④ $B_y=\frac{i}{(\frac{\omega}{c})^2-k^2}\left(k\frac{\partial B_z}{\partial y}+\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$ (iii) + (v)

Inserting into (i) - (iv) $\longrightarrow$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]E_z=0$$
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z=0$$

If $E_z=0\rightarrow$ TE waves

$B_z=0\rightarrow$ TM waves

If both $E_z, B_z=0$ TEM waves

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0$$

$\tilde{E}=-\nabla\phi$ $\nabla^2\phi=0$ → ϕ constant throughout

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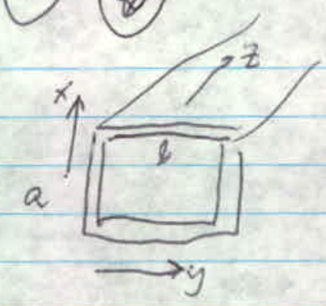
Solution for Rectangular Waveguide



TE waves

$\boxed{E_z = 0}$

$$B_z(x,y) = X(x)\,Y(y)$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]XY = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$$

$$-k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

~~B_x~~

Gen'l sol'n is $X = A\sin(k_x x) + B\cos(k_x x)$

But at $x=0$, & $x=a$, $B_x = 0 \rightarrow \frac{\partial B_z}{\partial x} = 0$

$$\frac{dX}{dx} = k_x A\cdot\cos(k_x x) - k_x B\sin(k_x x)$$

at $x=0$, $A=0$

$$k_x = \frac{m\pi}{a}$$

$$X = B\cos\left(\frac{m\pi x}{a}\right)$$

Similarly $Y = D\cos\left(\frac{n\pi y}{b}\right)$

$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$: TE_{mn} mode

$$k^2 = \left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$$

so if $\omega < c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \omega_{mn}$ — Low cutoff frequency

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}}$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{\left(\frac{dk}{d\omega}\right)} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c$$

Standing waves

Coax Transmission Line

$E_z = 0,\ B_z = 0$ possible (TEM wave)

From M.E. $k = \frac{\omega}{c}$

$cB_y = E_x, \quad cB_x = -E_y$

$\vec{E}, \vec{B}$ mutually $\perp$.

Together with $\nabla\cdot\vec{E} = 0$
$\nabla\cdot\vec{B} = 0$

gives $E_0(s,\phi) = \frac{A}{s}\hat{s}, \quad B_0(s,\phi) = \frac{A}{cs}\hat{\phi}$

$$E(s,\phi,z,t) = \frac{A\cos(kz-\omega t)}{s}\hat{s}$$

$$B(s,\phi,z,t) = \frac{A\cos(kz-\omega t)}{cs}\hat{\phi}$$

~~[illegible]~~ Group velocity

Conductivity & Im. part of $\epsilon(\omega)$

$$\nabla\times\vec{B}(\omega) = \epsilon(\omega)\mu(\omega)\frac{\partial\vec{E}(\omega)}{\partial t} = i\omega\left[\frac{\epsilon_b}{\epsilon_0}\right]\mu(\omega)\vec{E}(\omega) - \frac{Ne^2}{m}\frac{f_0}{(\gamma_0-i\omega)}\mu(\omega)\vec{E}(\omega)$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = \frac{\hat{\epsilon}_b}{\epsilon_0} + \frac{iNe^2 f_0}{\epsilon_0 m\omega(\gamma_0-i\omega)}$$

$$\nabla\times\vec{B}(\omega) = i\omega\mu(\omega)\epsilon(\omega)\frac{\partial E(\omega)}{\partial t} + \mu(\omega)\frac{Ne^2 f_0}{\epsilon_0 m(\gamma_0-i\omega)}E$$

$$= \mu(\omega)\epsilon(\omega)\frac{\partial E(\omega)}{\partial t} + \mu(\omega)\sigma(\omega)\vec{E}$$

$$\sigma = \frac{Ne^2 f_0}{m(\gamma_0-i\omega)} = \frac{Ne^2\tau}{m} \quad \left(\tau = \frac{f_0}{\gamma_0}\right) \simeq [illegible]\times 10^{-14}\,s$$

Vector Potentials

Maxwell's Eqs (i) $\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$ $\quad\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ (iii)

$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ (iv)

(ii) $\nabla\cdot\vec{B} = 0$

Since $\nabla\cdot\vec{B}=0$ we can write $\boxed{\vec{B} = \nabla\times\vec{A}}$ ($\vec{A}$ some vector)

Then $\nabla\times\vec{E} = -\frac{\partial}{\partial t}(\nabla\times A)$ or $\nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$

so $\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\nabla V$ (V some scalar)

so $\boxed{\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}}$ replaces electrostatics expression

Putting this Eqn into (i), $\longrightarrow$

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\frac{1}{\epsilon_0}\rho}$$ (replaces Poisson Eqn)

Putting expressions for $\vec{B}$ + $\vec{E}$ in boxes above into (M.E. IV) above

$$\nabla\times(\nabla\times\vec{A}) = \mu_0\vec{J} - \mu_0\epsilon_0\nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

→ using Expression $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

$$\longrightarrow \boxed{\left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0 J}$$

Gauge Transformations

V, + $\vec{A}$ are <u>not</u> unique!

Write $\vec{A}' = \vec{A} + \vec{\alpha}$ $\quad V' = V + \beta$

If $\nabla\times\vec{\alpha} = 0$ we get same $\vec{B}$, so $\alpha = \nabla\lambda$

If $\boxed{\nabla\beta + \frac{\partial\vec{\alpha}}{\partial t} = 0}$ we get same $\vec{E}$

or $\nabla\left(\beta + \frac{\partial\lambda}{\partial t}\right) = 0$

so $\beta + \frac{\partial\lambda}{\partial t} = k(t)$ independent of $\vec{r}$.

Define new $\lambda \to \lambda + \int_0^t k(t')\,dt'$

Then $\left.\begin{aligned}\vec{A}' &= A + \nabla\lambda\\ V' &= V - \frac{\partial\lambda}{\partial t}\end{aligned}\right\}$ gives same fields $\vec{B}, \vec{E}$ for any $\lambda(\vec{r}, t)$

This is known as a <u>Gauge Transformation</u>

<u>Magnetostatics</u> $\nabla \cdot \vec{A} = 0$ (<u>Coulomb Gauge</u>)

$$\longrightarrow \nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

$$V(\vec{r},t) = -\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}}\, d\vec{r}' \longrightarrow \text{determined by charge at } \underline{\text{same time}}.$$

Action at a distance? No! $\vec{E}$ involves $\frac{\partial \vec{A}}{\partial t}$ as well

$\Rightarrow \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ changes only after signal "arrives" at $\vec{r}$ from change of $\rho(\vec{r}',t)$.

Advantage of Coulomb gauge: V is simple to calculate

Disadvantage: $\vec{A}$ is <u>difficult</u> to calculate:

$$\nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 \vec{J} + \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right)$$

<u>Lorenz Gauge</u> $\nabla \cdot \vec{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$

$$\longrightarrow \nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\& \quad \nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho$$

V & $\vec{A}$ are treated on equal footing!

Can write $\boxed{\begin{aligned} \Box^2 V &= -\frac{1}{\epsilon_0}\rho \\ \Box^2 \vec{A} &= -\mu_0 \vec{J} \end{aligned}}$ ① $\quad \Box^2 \equiv \nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}$ (D'Alembertian operator)

<u>Lorentz Force</u>

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) = q\left[-\nabla V - \frac{\partial \vec{A}}{\partial t} + v\times(\nabla\times\vec{A})\right]$$

Now $\vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}$

$$\text{so} \quad \frac{d\vec{p}}{dt} = -q\left[\frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A} - \nabla(\vec{v}\cdot\vec{A}) + \nabla V\right]$$

$$= -q\left[\frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A} + \nabla(V - \vec{v}\cdot\vec{A})\right]$$

$\frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A} = \frac{d\vec{A}}{dt}$ total rate of change of $\vec{A}$ at moving particle so $\frac{d}{dt}(\vec{p}+q\vec{A}) = -\nabla\left[q(V-\vec{v}\cdot\vec{A})\right]$

Compare $\frac{d\vec{p}}{dt} = -\nabla U$

so $(\vec{p}+q\vec{A})$ is called <u>canonical momentum</u> P_{can}

& $U = q(V-\vec{v}\cdot\vec{A})$

$\vec{A}$ is a kind of "momentum/unit charge"

& V is potential energy/unit charge

<u>Retarded Potentials</u>

In static case, in Lorenz Gauge

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \qquad \nabla^2\vec{A} = -\mu_0\vec{J}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')\,d\vec{r}'}{\mathcal{r}} \qquad A(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')\,d\vec{r}'}{\mathcal{r}}$$

$$\mathcal{r} = |\vec{r}-\vec{r}'|$$

But since em waves travel at speed of light, $V(\vec{r})$, $A(\vec{r})$ for time-varying charges must be replaced by

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\rho\frac{(\vec{r}',t_r)}{\mathcal{r}}d\vec{r}' \qquad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\mathcal{r}}d\vec{r}'}$$

$t_r = t - \frac{\mathcal{r}}{c}$ $\longrightarrow$ <u>retarded potentials</u>

Retarded potentials satisfy M.E's (Eq. 1) + Lorenz condition

$$\nabla V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\nabla\rho\cdot\frac{1}{\mathcal{r}} + \rho\nabla\left(\frac{1}{\mathcal{r}}\right)\right]d\vec{r}'$$

$$\nabla\rho(\vec{r}',t_r) = \dot{\rho}\,\nabla t_r = -\frac{1}{c}\dot{\rho}\,\nabla\mathcal{r}$$

$$\nabla\mathcal{r} = \hat{\mathcal{r}}$$

$$\nabla\left(\frac{1}{\mathcal{r}}\right) = -\frac{\hat{\mathcal{r}}}{\mathcal{r}^2}$$

so $\nabla V = \frac{1}{4\pi\epsilon_0}\int\left(-\frac{\dot{\rho}}{c}\frac{\hat{\mathcal{r}}}{\mathcal{r}} - \rho\frac{\hat{\mathcal{r}}}{\mathcal{r}^2}\right)d\vec{r}'$

Taking Divergence $\longleftarrow$ $\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int -\frac{1}{c}\left[\frac{\hat{\mathcal{r}}}{\mathcal{r}}\cdot(\nabla\dot{\rho}) + \dot{\rho}\,\nabla\cdot\left(\frac{\hat{\mathcal{r}}}{\mathcal{r}}\right)\right] - \left[\frac{\hat{\mathcal{r}}}{\mathcal{r}^2}\cdot(\nabla\rho) + \rho\,\nabla\cdot\left(\frac{\hat{\mathcal{r}}}{\mathcal{r}^2}\right)\right]dr'$

But $\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\,\nabla r = -\frac{1}{c}\ddot{\rho}\,\hat{r}$

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta(\vec{r}) \quad \& \quad \nabla\cdot\left(\frac{\hat{r}}{r}\right) = \frac{1}{r^2}$$

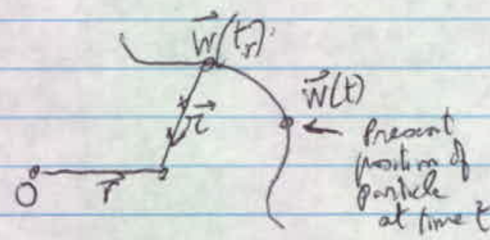
so $\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{r} - 4\pi\rho\,\delta(\vec{r})\right]d\tau' = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{\rho(\vec{r},t)}{\epsilon_0}$

Same can be shown for $\vec{A}$.

HW Problems 10.2, 10.4, 10.5, 10.7, 10.8

Lienard-Wiechert Potential (due to a single moving charge q)

Position of charge as fn of time $\vec{w}(t)$



$\vec{w}(t_r)$ is position of charge at time $t_{\mathscr{r}}$ for signal to get to pt. $\vec{r}$ at time t

$$t_r = t - \frac{\mathscr{r}}{c}$$

Then $V(\vec{r}, t) = \dfrac{q}{4\pi\epsilon_0}\displaystyle\int \frac{\delta(\vec{r}\,' - \vec{w}(t_r))}{\mathscr{r}}\, d\vec{r}\,'$ since $\rho(\vec{r}\,') = q\,\delta(\vec{r}\,' - \vec{w}(t_r))$

Cannot integrate directly over $\vec{r}\,'$ as delta-fn depends on $\vec{w}(t_r)$, which also depends on r' through $\frac{\mathscr{r}}{c}$.

So change variables: Write $\vec{r}\,'' = \vec{r}\,' - \vec{w}(t_r)$ (1)

Then $d\vec{r}\,'' = |J|\, d\vec{r}\,'$ where J is the Jacobian

$$|J| = \begin{Vmatrix} \frac{\partial x''}{\partial x'} & \frac{\partial x''}{\partial y'} & \frac{\partial x''}{\partial z'} \\ \frac{\partial y''}{\partial x'} & \frac{\partial y''}{\partial y'} & \frac{\partial y''}{\partial z'} \\ \frac{\partial z''}{\partial x'} & \frac{\partial z''}{\partial y'} & \frac{\partial z''}{\partial z'} \end{Vmatrix}$$

From (1) $\dfrac{\partial x''}{\partial x'} = 1 - \left(\dfrac{d\vec{w}}{dt}\right)_x \dfrac{\partial t_r}{\partial x'} = 1 - v_x\left(-\dfrac{1}{c}\dfrac{\partial \mathscr{r}}{\partial x'}\right)$

But $\dfrac{\partial \mathscr{r}}{\partial x'} = \dfrac{\partial}{\partial x'}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{\frac{1}{2}} = \dfrac{-(x-x')}{\mathscr{r}} = -\hat{\mathscr{r}}_x$

$$\text{so } \frac{\partial x''}{\partial x'} = 1 - \frac{v_x}{c}\hat{\mathscr{r}}_x$$

Similarly, $\dfrac{\partial x''}{\partial y'} = -\dfrac{v_x}{c}\hat{\mathscr{r}}_y$; $\dfrac{\partial x''}{\partial z'} = -\dfrac{v_x}{c}\hat{\mathscr{r}}_z$

$\dfrac{\partial y''}{\partial x'} = -\dfrac{v_y}{c}\hat{\mathscr{r}}_x$; $\dfrac{\partial y''}{\partial y'} = 1 - \dfrac{v_y}{c}\hat{\mathscr{r}}_y$; $\dfrac{\partial y''}{\partial z'} = -\dfrac{v_y}{c}\hat{\mathscr{r}}_z$

$\dfrac{\partial z''}{\partial x'} = -\dfrac{v_z}{c}\hat{\mathscr{r}}_x$, $\dfrac{\partial z''}{\partial y'} = -\dfrac{v_z}{c}\hat{\mathscr{r}}_y$, $\dfrac{\partial z''}{\partial z'} = 1 - \dfrac{v_z}{c}\hat{\mathscr{r}}_z$

We can choose x-axis along $\vec{v}$ instantaneously, so

$$|J| = \begin{Vmatrix} 1 - \frac{\vec{v}\cdot\hat{\mathscr{r}}}{c} & -\frac{v_x}{c}\hat{\mathscr{r}}_y & -\frac{v_x}{c}\hat{\mathscr{r}}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} = \left[1 - \frac{\vec{v}\cdot\hat{\mathscr{r}}}{c}\right]$$

$$\text{so} \quad d\vec{r}' = \frac{1}{|J|} d\vec{r}'' = \frac{1}{\left[1 - \frac{\vec{v}\cdot\hat{\imath}}{c}\right]} d\vec{r}''$$

Then integral can be written

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\vec{\imath}'')\; d\vec{r}''}{\left[1 - \frac{\vec{v}\cdot\hat{\imath}}{c}\right] \imath} = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(1 - \frac{\vec{v}\cdot\hat{\imath}}{c}\right)\imath}$$

where $\vec{\imath}$ satisfies $\left[\vec{r} - \vec{\imath} - \vec{w}(t - \frac{\imath}{c}) = 0\right]$

or $\boxed{\vec{\imath} = \vec{r} - \vec{w}(t_r)}$

Can write $V(\vec{r},t)$ above as

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{q c}{(\imath c - \vec{\imath}\cdot\vec{v})}}$$

Similarly, we can show that

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)\, d\vec{r}'}{\imath} = \frac{\mu_0}{4\pi} \vec{v}(t_r) \int \frac{\rho(\vec{r}', t_r)\, d\vec{r}'}{\imath}$$

(since $\vec{J} = \rho \vec{v}$)

$$= \frac{\mu_0}{4\pi} \vec{v}\, (4\pi\epsilon_0)\, V(\vec{r},t)$$

$$\boxed{\vec{A}(\vec{r},t) = (\mu_0\epsilon_0)\, \vec{v}\, V(\vec{r},t) = \frac{\vec{v}}{c^2} V(\vec{r},t)}$$

Example: Point charge moving with constant velocity $\vec{v}$.

Let $\vec{w}(t) = \vec{v}t$ (at origin at time $t=0$)

$\vec{\imath}$ satisfies $\vec{\imath} = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}t_r$

$+\ t_r = t - \frac{\imath}{c}$ so $\imath = c(t - t_r)$

$$\rightarrow \quad \imath^2 = c(t - t_r)^2$$

$$\text{or} \quad (\vec{r} - \vec{v}t_r)^2 = c(t - t_r)^2$$

solve for t_r in terms of $\vec{r}, t$ $\rightarrow \quad r^2 - 2\vec{r}\cdot v\, t_r + v^2 t_r^2 = c^2t^2 - 2c^2 t t_r + c^2 t_r^2$

$$\rightarrow \quad (c^2 - v^2)t_r^2 + (2\vec{r}\cdot\vec{v} - 2c^2t)t_r + (c^2t^2 - r^2) = 0$$

$$\rightarrow \quad t_r = \frac{c^2t - \vec{r}\cdot\vec{v} \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{(c^2 - v^2)}$$

Consider $v = 0$, $\quad t_r = t \pm \frac{r}{c}$ but it must be $t - \frac{r}{c}$, so

choose negative sign

$$\text{so} \quad t_r = \frac{(c^2t - \vec{r}\cdot\vec{v}) - \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

so $V(\vec{r},t)$ for a point charge moving with uniform velocity $\vec{v}$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathfrak{r}c - \vec{\mathfrak{r}}\cdot\vec{v})}$$

$$\mathfrak{r} = c(t - t_r) \qquad \vec{\mathfrak{r}} = \vec{r} - \vec{v}t_r$$

$$\begin{aligned}
\mathfrak{r}c - \vec{\mathfrak{r}}\cdot\vec{v} &= c^2(t - t_r) - \vec{r}\cdot\vec{v} + v^2 t_r \\
&= (c^2 t - \vec{r}\cdot v) - (c^2 - v^2)t_r \\
&= \cancel{(c^2 t - \vec{r}\cdot\vec{v})} - \cancel{c^2 t} + \cancel{\vec{r}\cdot v} + \sqrt{(c^2 t - r^2 v)^2 + (c^2 - v^2)(r^2 - c^2 t^2)}
\end{aligned}$$

$$\text{so} \quad V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\sqrt{(c^2 t - r^2 v)^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}$$

$$A(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{\sqrt{\cdots}}$$

To evaluate $V(\vec{r},t)$ we need to calculate $(\mathcal{r}c - \vec{\mathcal{r}}\cdot\vec{v})$

$$\vec{\mathcal{r}} = \vec{r} - \vec{v}t_r \qquad \mathcal{r} = c(t-t_r)$$

$$\Rightarrow \mathcal{r}c - \vec{\mathcal{r}}\cdot\vec{v} = c^2(t-t_r) - \vec{r}\cdot\vec{v} + v^2 t_r = \left[c^2 t - \vec{v}\cdot\vec{r} - (c^2-v^2)t_r\right]$$

substituting for t_r from above, we get

$$\mathcal{r}c - \vec{\mathcal{r}}\cdot\vec{v} = \sqrt{(c^2t^2 - \vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}$$

$$\therefore V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t^2-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}}$$

$$\& \ A(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{\sqrt{\cdots\cdots}}$$

Field of an arbitrarily moving point charge:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathcal{r}c - \vec{\mathcal{r}}\cdot\vec{v})} \qquad A(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\nabla V = -\frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathcal{r}c-\vec{\mathcal{r}}\cdot\vec{v})^2}\nabla(\mathcal{r}c - \vec{\mathcal{r}}\cdot\vec{v})$$

$$\vec{\mathcal{r}} = \vec{r} - \vec{w}(t) \quad \vec{v} = \dot{\vec{w}}(t_r)$$

$$\mathcal{r} = c(t-t_r) \longrightarrow \nabla\mathcal{r} = -c\nabla t_r$$

$$\nabla(\vec{\mathcal{r}}\cdot\vec{v}) = (\vec{\mathcal{r}}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{\mathcal{r}} + \vec{\mathcal{r}}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{\mathcal{r}})$$

$$(\vec{\mathcal{r}}\cdot\nabla)\vec{v} = \left(\mathcal{r}_x\frac{\partial}{\partial x} + \mathcal{r}_y\frac{\partial}{\partial y} + \mathcal{r}_z\frac{\partial}{\partial z}\right)\vec{v}(t_r)$$

$$= \mathcal{r}_x\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathcal{r}_y\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathcal{r}_z\frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial z}$$

$$= \vec{a}(\vec{\mathcal{r}}\cdot\nabla t_r) \qquad (\vec{a} = \text{acceleration of particle} = \frac{d\vec{v}}{dt_r})$$

$$(\vec{v}\cdot\nabla)\vec{\mathcal{r}} = (\vec{v}\cdot\nabla)\vec{r} - \vec{v}\cdot\nabla(\vec{w})$$

$$(\vec{v}\cdot\nabla)\vec{r} = \vec{v} \quad \text{(see book)}$$

$$(\vec{v}\cdot\nabla)\vec{w} = \vec{v}(\vec{v}\cdot\nabla t_r)$$

$$\nabla\times\vec{v} = -\vec{a}\times\nabla t_r \quad \text{(see book)}$$

$$\nabla\times\vec{\mathcal{r}} = \underbrace{\cancel{\nabla\times\vec{r}}}_{0} - \nabla\times\vec{w} = -\vec{v}\times\nabla t_r$$

$$\longrightarrow \nabla(\vec{\mathcal{r}}\cdot\vec{v}) = \vec{v} + (\vec{\mathcal{r}}\cdot\vec{a} - v^2)\nabla t_r$$

$$\longrightarrow \nabla V = \frac{qc}{4\pi\epsilon_0(\mathcal{r}c - \vec{\mathcal{r}}\cdot\vec{v})^2}\left[\vec{v} + (c^2 - v^2 + \vec{\mathcal{r}}\cdot\vec{a})\nabla t_r\right]$$

Now $\mathscr{r} = c(t - t_r)$

$-c\nabla t_r = \nabla \mathscr{r}$

(see book) we obtain $\nabla t_r = \dfrac{-\vec{\mathscr{r}}}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})}$

so $\nabla V = \dfrac{1}{4\pi\epsilon_0}\dfrac{qc}{(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec v)^3}\left[(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec v)\vec v - (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec a)\vec{\mathscr{r}}\right]$

Similarly $\dfrac{\partial \vec A}{\partial t} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec v)^3}\left[(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec v)(-\vec v + \mathscr{r}\vec a/c) + \dfrac{\mathscr{r}}{c}(c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec a)\vec v\right]$

Define $\vec u = c\hat{\mathscr{r}} - \vec v$

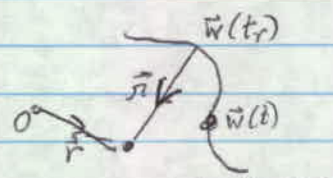
and we get $\boxed{\vec E(\vec r,t) = \dfrac{q}{4\pi\epsilon_0}\dfrac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec u)^3}\left[(c^2-v^2)\vec u + \overbrace{\vec{\mathscr{r}}\times(\vec u\times\vec a)}^{(\vec{\mathscr{r}}\cdot\vec a)\vec u-(\vec{\mathscr{r}}\cdot\vec u)\vec a}\right]}$

$\nabla\times\vec A = \dfrac{1}{c^2}[\nabla\times(V\vec v)] = \dfrac{1}{c^2}\left[V\underbrace{(\nabla\times\vec v)}_{\text{calculated previously}} - \vec v\times\underbrace{\nabla V}_{\text{calculated previously}}\right]$

so $\nabla\times\vec A = -\dfrac{1}{c}\dfrac{q}{4\pi\epsilon_0}\dfrac{1}{(\vec u\cdot\vec{\mathscr r})^3}\vec{\mathscr r}\times\left[(c^2-v^2)\underbrace{\vec v}_{\text{can write as } -\vec u} + (\vec{\mathscr r}\cdot\vec a)\vec v + (\vec{\mathscr r}\cdot\vec u)\vec a\right]$

But $\vec B = \nabla\times\vec A$

so $\boxed{\vec B(\vec r,t) = \dfrac1c \hat{\mathscr r}\times\vec E(\vec r,t)}$



In $\vec E(\vec r,t)$ first term $\sim \dfrac{1}{\mathscr r^2}$ → known as generalized Coulomb Field

second term $\sim \dfrac1{\mathscr r}$ so is dominant at large distances. (known as radiation field)

Example:

Point charge moving with constant velocity

$\vec a = 0$ $\quad \vec E = \dfrac{q}{4\pi\epsilon_0}\dfrac{(c^2-v^2)\mathscr r\,\vec u}{(\vec{\mathscr r}\cdot\vec u)^3}$ $\quad \vec w = \vec v t$

$\mathscr r\vec u = c\vec{\mathscr r} - \mathscr r\vec v = c(\vec r - \vec v t_r) - c(t-t_r)\vec v = c(\vec r - \vec v t)$

$(\vec{\mathscr r}\cdot\vec u) = \mathscr r c - \vec{\mathscr r}\cdot\vec v = \sqrt{(c^2t - \vec r\cdot\vec v)^2 + (c^2-v^2)(r^2-c^2t^2)}$

$\dfrac{t^4}{T^2}$ $\quad \dfrac{t^2 2^2}{T^2}$

$$(\vec{\mathscr{r}}\cdot\vec{u}) = Rc\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}$$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{c^2\left(1-\frac{v^2}{c^2}\right)\mathscr{r}\vec{u}}{R^3c^3\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}} = \frac{q}{4\pi\epsilon_0}\frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}\frac{\hat{R}}{R^2}$$

$$\left[\mathscr{r}\vec{u} = c(\vec{r}-\vec{v}t) = c\vec{R}\right]$$

$$\cancel{\vec{B} = \frac{1}{c}(\hat{\mathscr{r}}\times\vec{E})} =$$

$$\hat{\mathscr{r}} = \frac{\vec{r}-\vec{v}t_r}{\mathscr{r}} = \frac{(\vec{r}-\vec{v}t) + (t-t_r)\vec{v}}{\mathscr{r}} = \frac{\vec{R}}{\mathscr{r}} + \frac{\vec{v}}{c}$$

$$\vec{B} = \frac{1}{c}\left(\hat{\mathscr{r}}\times\vec{E}\right) = \frac{1}{c}\left[\cancel{\frac{\vec{R}}{\mathscr{r}}\times\vec{E}}\right] + \frac{1}{c^2}\left(\vec{v}\times\vec{E}\right)$$

$$\vec{E} \parallel \vec{R}$$

If $v \ll c$

$$\vec{E}(\vec{r},t) \longrightarrow \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2} \qquad B(\vec{r},t) \longrightarrow \frac{\mu_0}{4\pi}\frac{q}{R^2}(\vec{v}\times\hat{R})$$

<u>Radiated Power</u> Surface of radius r around source ($r \gg$ Dimensions of source)

$$P(r,t) = \int_S \vec{S}\cdot d\vec{a} = \frac{1}{\mu_0}\int_S(\vec{E}\times\vec{B})\cdot d\vec{a}$$

$$P_{rad}(t_0) = \lim_{r\to\infty} P\left(r, t_0+\frac{r}{c}\right)$$

only <u>acceleration</u> terms give non-vanishing radiation fields

<u>HW 4</u> 10.11, 10.12, 10.15, 10.16, 10.19, 10.20