

# PHYS 201 Mathematical Physics, Fall 2016, Homework 6

**Due date: Tuesday, November 29th, 2016**

1. Use integration by parts and show that the asymptotic expansion as  $x \rightarrow \infty$  of the upper incomplete Gamma function  $\Gamma(1/2, x)$  is

$$\begin{aligned}\Gamma(1/2, x) &= \int_0^x t^{-1/2} e^{-t} dt \\ &\sim \Gamma(1/2) - \frac{e^{-x}}{\sqrt{x}} \left[ 1 - \frac{1}{2x} + \dots \right]\end{aligned}$$

2.

- a. Use Laplace's method to verify the first two or three terms of the asymptotic expansion as  $x \rightarrow \infty$  of the modified Bessel function  $K_0(x)$

$$\begin{aligned}K_0(x) &= \int_1^\infty (s^2 - 1)^{-1/2} e^{-xs} ds \\ &\sim \frac{e^{-x}}{\sqrt{x}} \sum_{n=0}^{\infty} (-1)^n \frac{[\Gamma(n + \frac{1}{2})]^2}{2^{n+1/2} n! \Gamma(\frac{1}{2}) x^n}\end{aligned}$$

This is an application of Watson's lemma where the integral  $\int_0^{a>0} h(t) e^{-xt} dt$  is approximated as  $x \rightarrow \infty$  by expanding  $h(t)$  in a power series about 0. (**Hint:** First shift the limits of the integral to 0 and  $\infty$ .)

- b. Use similar ideas as part (a) to find the first two terms of the asymptotic expansion as  $x \rightarrow \infty$  of

$$f(x) = \int_0^{\pi/2} e^{-x \sin^2 t} dt \tag{1}$$

- c. Show the following leading behavior as  $x \rightarrow \infty$  using Laplace's method:

$$\begin{aligned}f(x) &= \int_0^\infty \exp\left(-t - \frac{x}{\sqrt{t}}\right) dt \\ &\sim \pi^{1/2} 2^{2/3} 3^{-1/2} x^{1/3} e^{-3(x/2)^{2/3}}\end{aligned}$$

(**Hint:** Direct application of Laplace's method does not work (why?) Use the transformation  $t = sx^{2/3}$  and then apply Laplace's method.)

3. Use the method of stationary phase to find the leading behavior of the following integrals as  $x \rightarrow \infty$

i.  $\int_0^1 e^{ixt^2} \cosh t^2 dt$

ii.  $\int_{-1}^1 \sin[x(t - \sin t)] \sinh(t) dt$

4. Use the method of steepest descent to find the first two terms of the asymptotic expansion for the Airy function  $I(x)$  as  $x \rightarrow \infty$  where

$$I(x) = \int_{-\infty}^{\infty} \exp \left[ ix \left( \frac{t^3}{3} + t \right) \right] dt$$

See the full problem description in Exercise 2 of Section 6-3 of Carrier et al for hints.