PHYS 201 Mathematical Physics, Fall 2016, Homework 3

Due date: Tuesday, October 18th, 2016

1. Find the Taylor series expansions around the indicated points z_0. Where the function is multi-valued, give the results for at least two branches.

i. $z^{1/2}; z_0 = 1, i\pi$

ii. $(z - \pi)/(\sin z); z_0 = \pi$

2. Find the Laurent series expansion of

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

in the three regions, $|z| < 1$, $1 < |z| < 2$ and $|z| > 2$.

3. In this exercise, we will find the solution (i.e., the complex potential Ω) to a Dirichlet problem inside the unit circle $|z| < 1$ with $\mathrm{Re}(\Omega) \equiv \phi = 0$ on the upper semicircle Γ_1 of the domain $(|z| = 1, \mathrm{Im}(z) > 0)$ and $\phi = k$ (with k real) on the lower semicircle Γ_2.

i. Recall the mapping $\zeta = f(z)$ from the unit circle to the infinite horizontal strip from Homework 1. Where do Γ_1 and Γ_2 lie after the transformation to ζ-space?

ii. Find the solution to the Dirichlet problem in the ζ-space. Now, find the solution in the original z space by substituting $\zeta = f(z)$. (**Hint:** If stuck, see Example 1 in Chapter 4 of Carrier et al)

iii. Verify that this solution is the same as the one obtained using Poisson's formula.

4. *Schwarz's lemma*: Let $f(z)$ be analytic inside the unit circle, with $f(0) = 0$, and with $|f(z)| \leq 1$ inside and on the circle. Show that $|f(z)| \leq |z|$ for any point z inside the circle and that if equality holds at any interior point then $f(z) = e^{i\alpha} z$ everywhere, with α some real constant. (**Hint:** The lemma can be proved in at least two ways. The first one is to construct some function $g(z)$ (you have to figure it out!) and use the maximum modulus theorem on g. The starting point for a second proof is to observe (show this), by writing the Taylor series for f about 0, that $\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = |a_0|^2 + |a_1|^2 + \ldots$, where $a_0, a_1, \ldots$ are the coefficients of the Taylor expansion of f. What can you say about the coefficients? Full points for giving one proof and bonus points for proving it in two different ways.)