Lecture 2: probability concepts II.

Laws of Probability

"There is this thing called *probability*. It obeys the laws of an axiomatic system. When identified with the real world, it gives (partial) information about the future."

- What axiomatic system?
- How to identify to real world?
  - Bayesian or frequentist viewpoints are somewhat different "mappings" from axiomatic probability theory to the real world
  - yet both are useful

"*And, it gives a consistent and complete calculus of inference.*"

Kolmogorov: axioms of probability theory and the Bayesian viewpoint

Kolmogorov probability concept

(Ω, F, P) probability space:

- sample space Ω (set of all possible outcomes)
- set of events F
- each event is a subset of Ω containing zero or more outcomes
- probability measure P: probability of some event A is P(A)

Axioms: (satisfied by frequentist definition of probabilities)

I. $P(A) \geq 0$ for an event A
II. $P(\Omega) = 1$ where Ω is the set of all possible outcomes
III. if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

disjoint

union of mutually exclusive

Example of a theorem:

Theorem: $P(\emptyset) = 0$
Proof: $A \cap \emptyset = \emptyset$, so
$P(A) = P(A \cup \emptyset) = P(A) + P(\emptyset)$, q.e.d.

Kolmogorov probability concept

Simple example: coin toss

Consider a single coin-toss, and assume that the coin will either land heads (H) or tails (T) (but not both). No assumption is made as to whether the coin is fair.

We may define:

$$\Omega = \{H, T\}$$
$$F = \{\varnothing, \{H\}, \{T\}, \{H, T\}\}$$

Kolmogorov's axioms imply that:

$$P(\varnothing) = 0$$

The probability of *neither* heads *nor* tails, is 0.

$$P(\{H, T\}) = 1$$

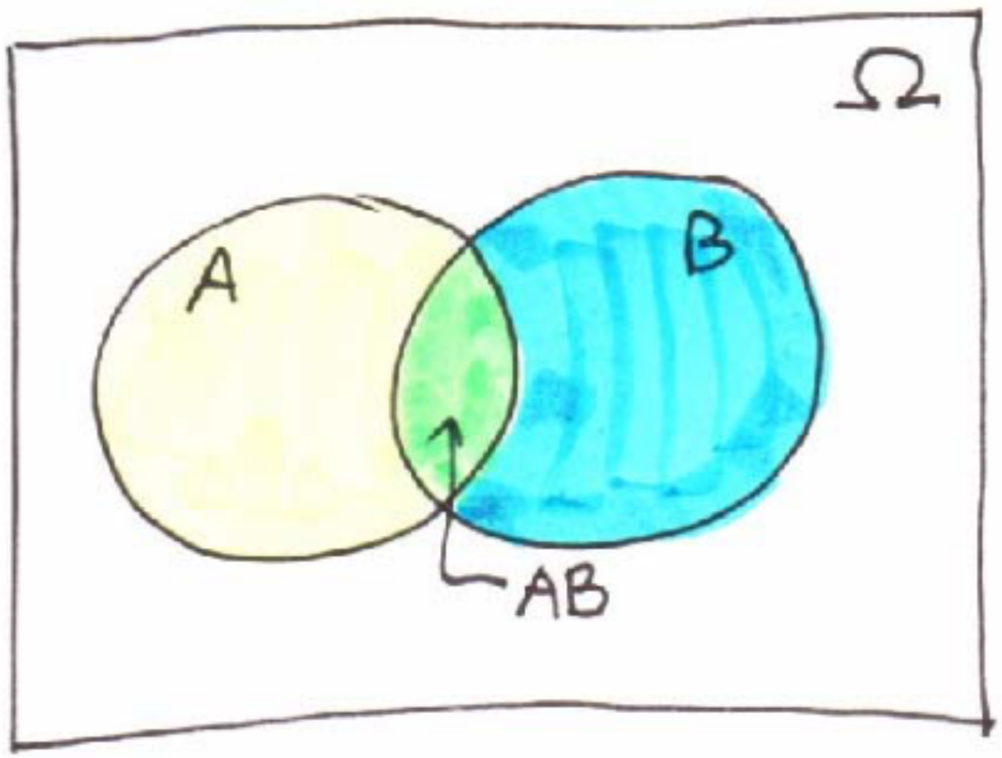
The probability of *either* heads *or* tails, is 1.

$$P(\{H\}) + P(\{T\}) = 1$$

The sum of the probability of heads and the probability of tails, is 1

Kolmogorov probability concept

Additivity or "Law of Or-ing"



Venn diagrams at web site of Probability, Mathematical Statistics, Stochastic Processes:

http://www.math.uah.edu/stat/

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

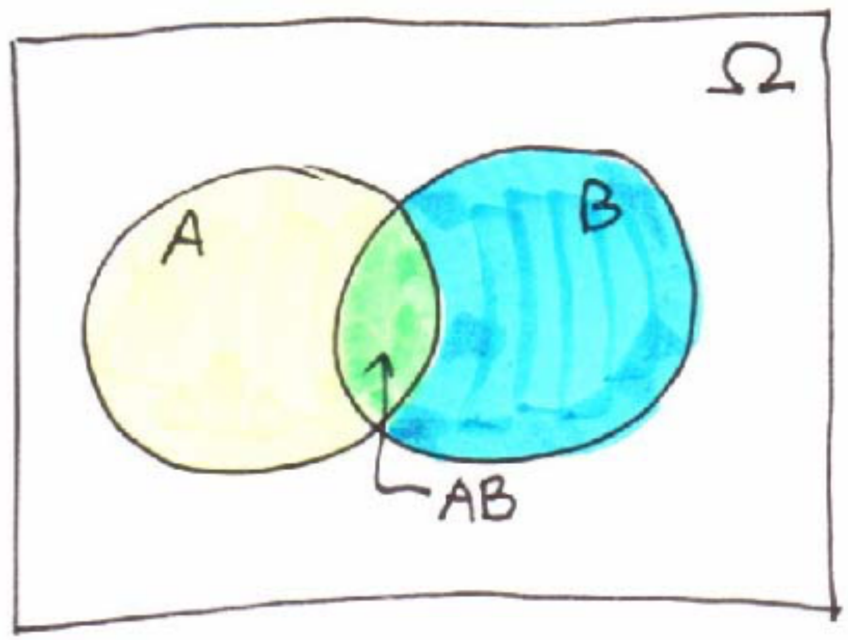
A or B

A and B

$$P(A \cap B)$$

Kolmogorov probability concept

Additivity or “Law of Or-ing”



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B \setminus (A \cap B)) \quad \text{(by Axiom 3)}$$

$$P(B) = P(B \setminus (A \cap B)) + P(A \cap B)\,.$$

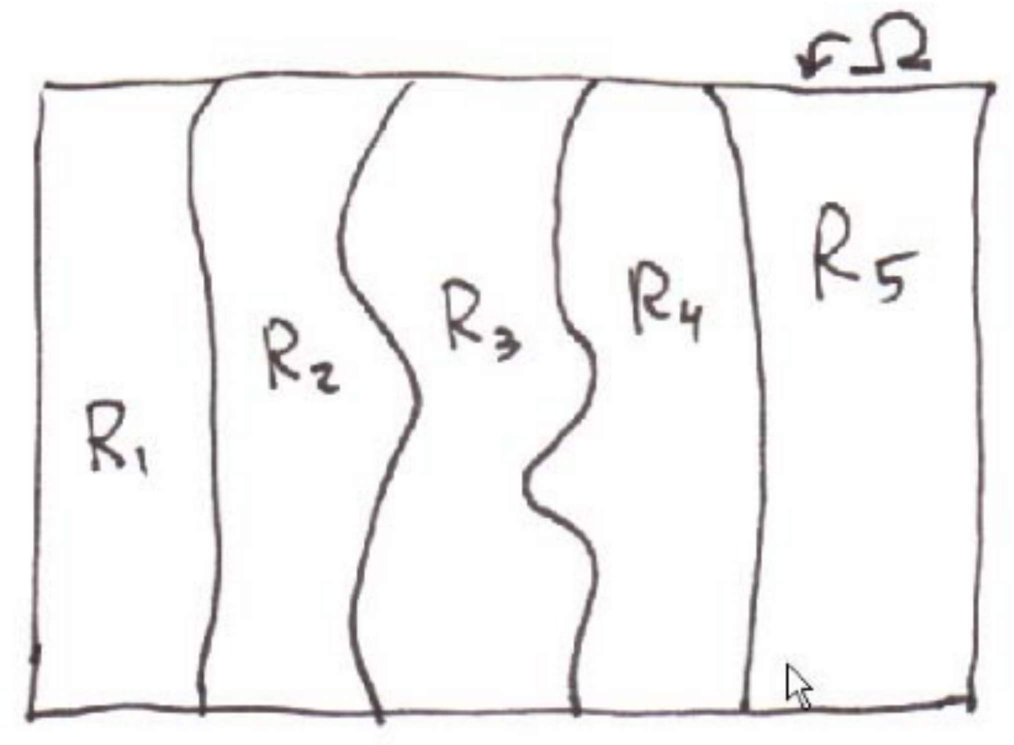
Eliminating $P(B \setminus (A \cap B))$ from both equations gives us the desired result.

This can be extended to the inclusion-exclusion principle

$$P\left(E^c\right) = P(\Omega \setminus E) = 1 - P(E)$$

Kolmogorov probability concept

"Law of Exhaustion"



If R_i are exhaustive and mutually exclusive (EME)

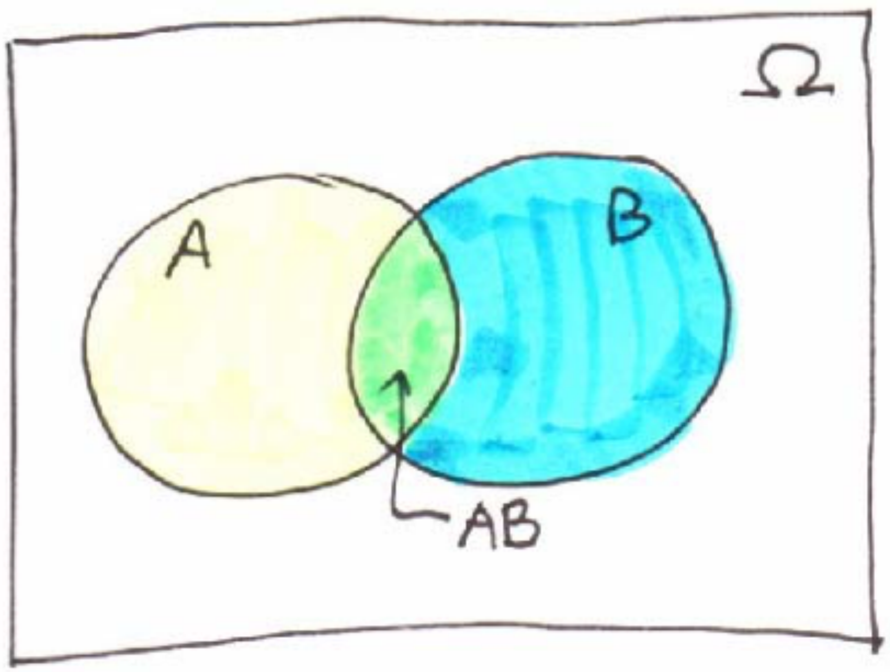
$$\sum_i P(R_i) = 1$$

This can be extended to the inclusion-exclusion principle

$$P\left(E^c\right) = P(\Omega \setminus E) = 1 - P(E)$$

Kolmogorov probability concept

Multiplicative Rule or "Law of And-ing"



(same picture as before)

"given"

$$P(AB) = P(A)P(B|A) = P(B)P(A|B)$$

$$P(B|A) = \frac{P(AB)}{P(A)}$$

"conditional probability"

"renormalize the outcome space"

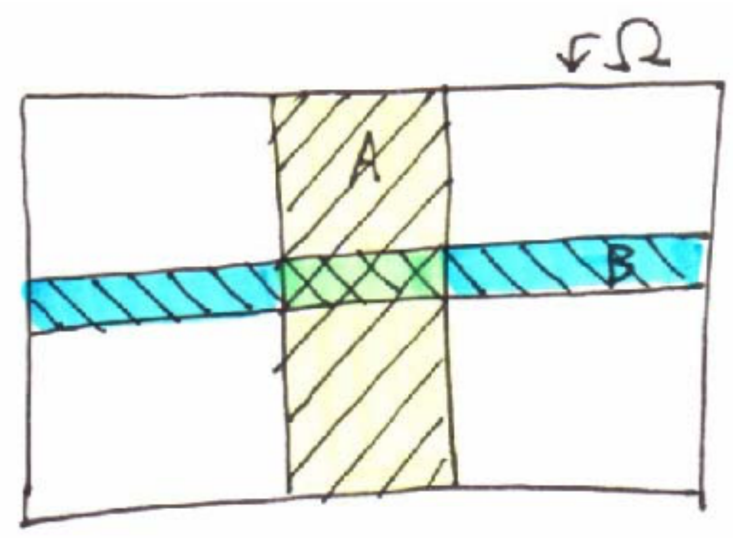
Kolmogorov probability concept

Similarly, for multiple And-ing:

$$P(ABC) = P(A)P(B|A)P(C|AB)$$

Independence:

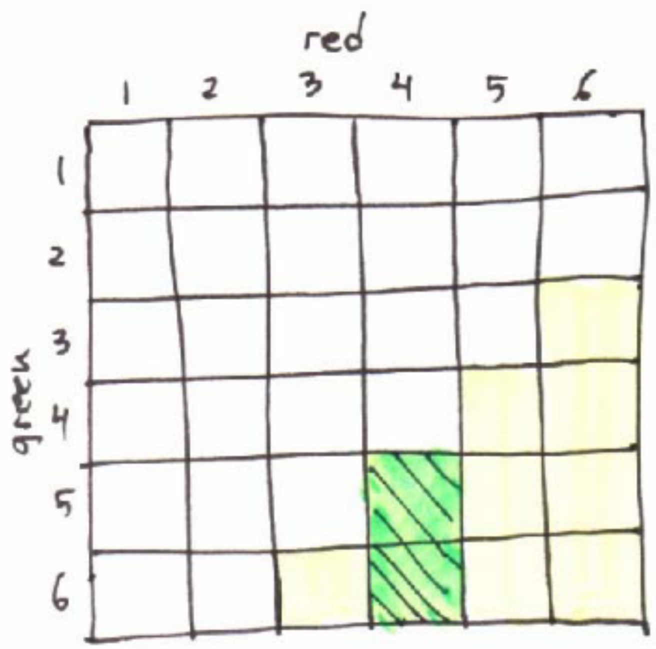
Events A and B are independent if
$P(A|B) = P(A)$
so $P(AB) = P(B)P(A|B) = P(A)P(B)$



Kolmogorov probability concept

A symmetric die has
$P(1) = P(2) = \ldots = P(6) = \frac{1}{6}$
Why? Because $\sum_i P(i) = 1$ and $P(i) = P(j)$.
Not because of "frequency of occurence in N trials".
That comes later!

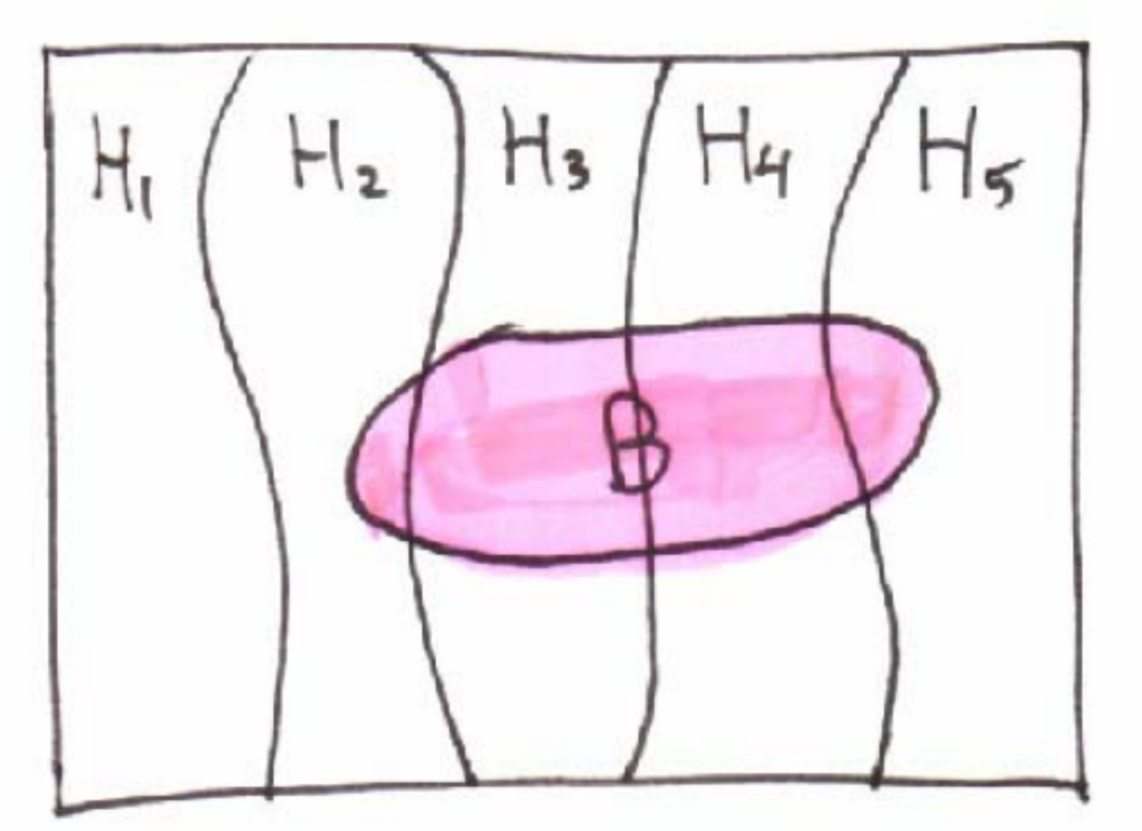
The sum of faces of two dice (red and green) is > 8.
What is the probability that the red face is 4?



$$P(R4\,|\,{>}8) = \frac{P(R4 \cap {>}8)}{P({>}8)} = \frac{2/36}{10/36} = 0.2$$

Kolmogorov probability concept

Law of Total Probability or “Law of de-Anding”



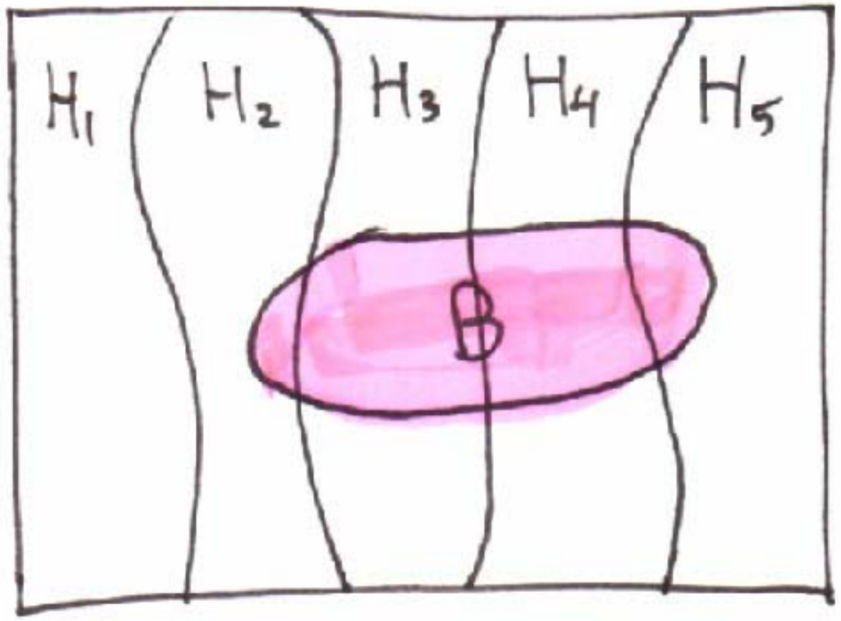
H’s are exhaustive and mutually exclusive (EME)

$$P(B) = P(BH_1) + P(BH_2) + \ldots = \sum_i P(BH_i)$$

$$P(B) = \sum_i P(B|H_i)P(H_i)$$

Bayes' theorem

Bayes Theorem



Thomas Bayes
1702 - 1761

(same picture as before)

Law of And-ing

$$P(H_i|B) = \frac{P(H_iB)}{P(B)}$$

$$= \frac{P(B|H_i)P(H_i)}{\sum_j P(B|H_j)P(H_j)}$$

Law of de-Anding

We usually write this as

$$P(H_i|B) \propto P(B|H_i)P(H_i)$$

this means, "compute the normalization by using the completeness of the H_i's"

Bayes' theorem

- As a theorem relating probabilities, Bayes is unassailable
- But we will also use it in inference, where the H's are hypotheses, while B is the data
  - "what is the probability of an hypothesis, given the data?"
  - some (defined as frequentists) consider this dodgy
  - others (Bayesians like us) ? consider this fantastically powerful and useful
  - in real life, the "war" between Bayesians and frequentists is long since over, and most statisticians adopt a mixture of techniques appropriate to the problem
    - for a view of the "war", see Efron paper on the course web site
- Note that you generally have to know a complete set of EME hypotheses to use Bayes for inference
  - perhaps its principal weakness

Bayes' theorem

Let's work a couple of examples using Bayes Law:

Example: Trolls Under the Bridge

Trolls are bad. Gnomes are benign.
Every bridge has 5 creatures under it:

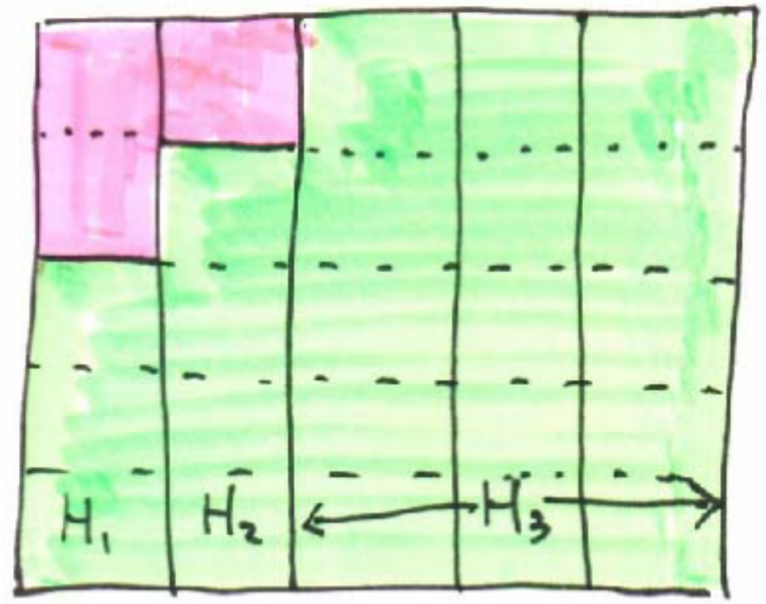
20% have TTGGG (H_1)
20% have TGGGG (H_2)
60% have GGGGG (benign) (H_3)

Before crossing a bridge, a knight captures one of the 5 creatures at random. It is a troll. "I now have an 80% chance of crossing safely," he reasons, "since only the case

20% had TTGGG (H1) → now have TGGG

is still a threat."

Bayes' theorem



$$P(H_i|T) \propto P(T|H_i)P(H_i)$$

so, $$P(H_1|T) = \frac{\frac{2}{5}\cdot\frac{1}{5}}{\frac{2}{5}\cdot\frac{1}{5}+\frac{1}{5}\cdot\frac{1}{5}+0\cdot\frac{3}{5}} = \frac{2}{3}$$

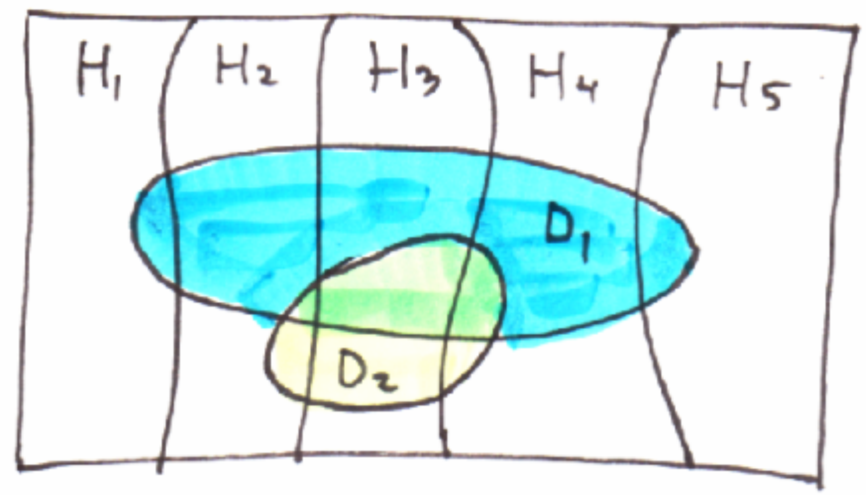
The knight's chance of crossing safely is actually only 33.3%
Before he captured a troll ("saw the data") it was 60%.
Capturing a troll actually made things worse!
(80% was never the right answer!)

Data changes probabilities!
Probabilities after assimilating data are called <u>posterior probabilities</u>.

Bayes' theorem

Commutivity/Associativity of Evidence

$P(H_i|D_1D_2)$ desired



We see D_1:

$P(H_i|D_1) \propto P(D_1|H_i)P(H_i)$

Then, we see D_2:

$P(H_i|D_1D_2) \propto P(D_2|H_iD_1)P(H_i|D_1)$ ← this is now a prior!

But,

$= P(D_2|H_iD_1)P(D_1|H_i)P(H_i)$

$= P(D_1D_2|H_i)P(H_i)$

this being symmetrical shows that we would get the same answer regardless of the order of seeing the data

All priors $P(H_i)$ are actually $P(H_i|D)$, conditioned on previously seen data! Often write this as $P(H_i|I)$. ← background information

Bayes' theorem

Bayes Law is a "calculus of inference", better (and certainly more self-consistent) than folk wisdom.

Example: Hempel's Paradox

Folk wisdom: A case of a hypothesis adds support to that hypothesis.

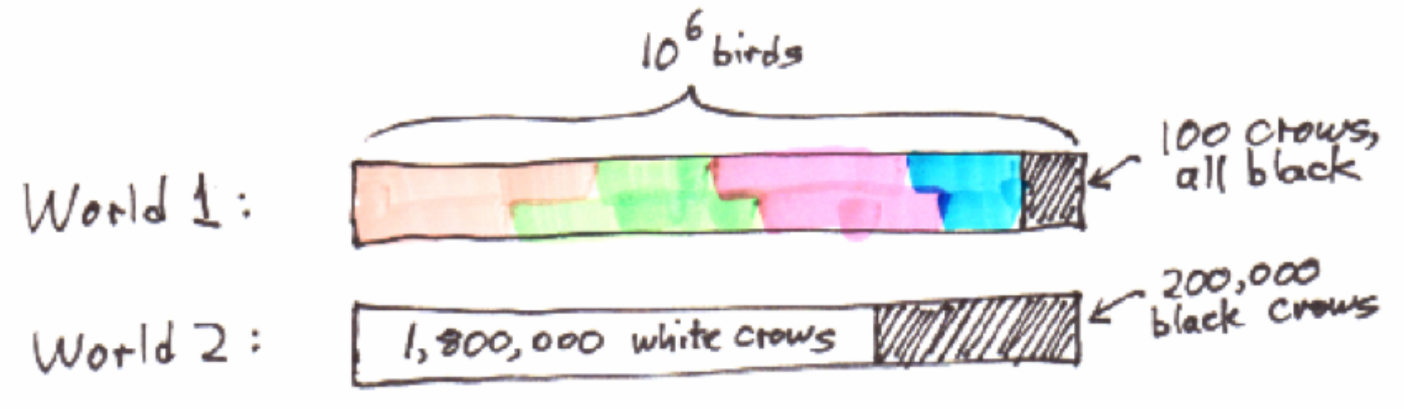
Example: "All crows are black" is supported by each new observation of a black crow.

All crows are black	$\Leftrightarrow$	All non-black things are non-crows

But this is supported by the observation of a white shoe.

So, the observation of a white shoe is thus evidence that all crows are black!

Bayes' theorem



I.J. Good: "The White Shoe is a Red Herring" (1966)

We observe one bird, and it is a black crow.
a) Which world are we in?
b) Are all crows black?

Important concept, Bayes odds ratio:

$$\frac{P(H_1|D)}{P(H_2|D)} = \frac{P(D|H_1)P(H_1)}{P(D|H_2)P(H_2)}$$

$$= \frac{0.0001\,P(H_1)}{0.1\,P(H_2)} = 0.001\frac{P(H_1)}{P(H_2)}$$

So the observation strongly supports H2 and the existence of white crows.

Hempel's folk wisdom premise is not true.

Data supports the hypotheses in which it is more likely compared with other hypotheses. (This is Bayes!)

We must have <u>some</u> kind of background information about the universe of hypotheses, otherwise data has no meaning at all.