Final project

Monte Carlo applications

problem 1: 10 % problem 2: 25%

due: December 9, 2016 3:00 pm

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Problem 1 submitted individually

Consider the probability distribution p(x) which is the mixture of two multivariate Gaussian distributions in two variables with $\mathbf{x}=(x_1,x_2)$:

$$p(\boldsymbol{x}) = \frac{1}{2}\mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) + \frac{1}{2}\mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$$

$$\text{where } \boldsymbol{\mu}_1 = [0,0]^T, \boldsymbol{\mu}_2 = [5,5]^T, \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \text{diag}\{0.25, 2\}$$

1(A) Plot the p(**x**) probability density function in the (x_1,x_2) variables

1(B) Calculate the mean of the vector $\mathbf{x}=(x_1,x_2)$ using Markov Chain Monte Carlo with Metropolis importance sampling. Compare the histogram with the **1a** plot.

1(C) Calculate the Monte Carlo error of the mean of the vector $\mathbf{x}=(x_1,x_2)$ using Markov Chain Monte Carlo with Metropolis importance sampling.

1(D) Estimate the autocorrelation time (separation of independent MC configurations) for correct error estimates.

1(E) Compare the MC results with the analytic expectations.

Problem 1 (phys 239 only) submitted individually

1(F) Compare the Metropolis MC results with Gibbs sampling.

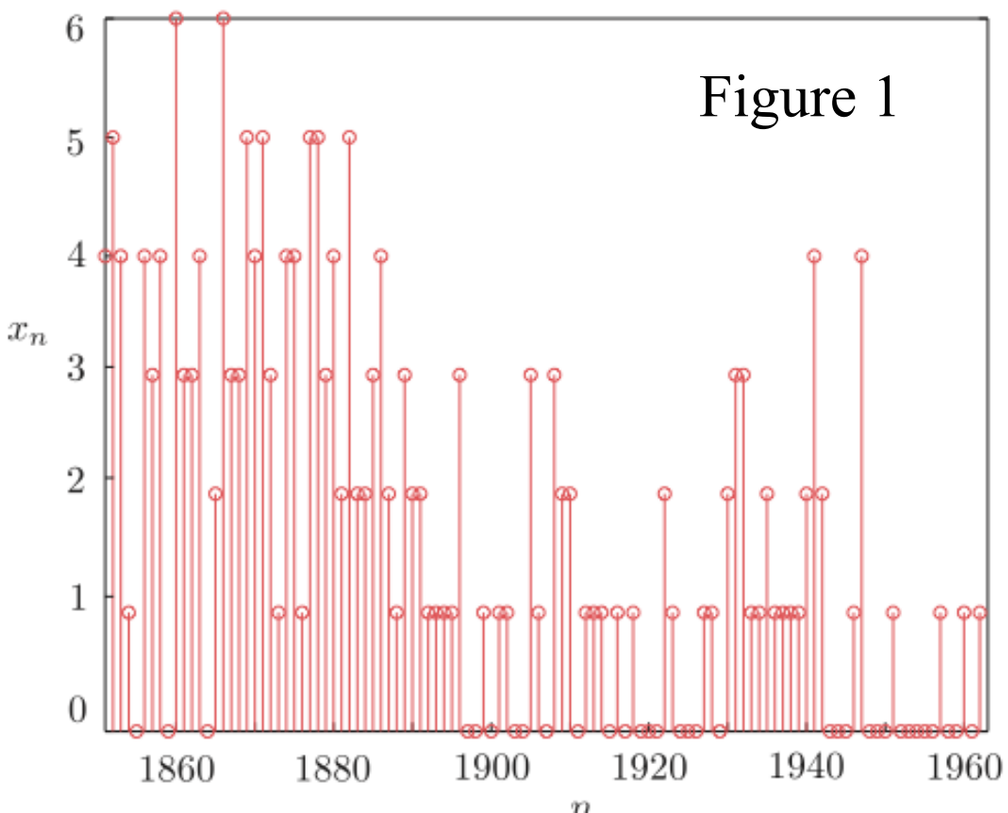
Problem 2 submitted by the team

A CASE STUDY: CHANGE-POINT DETECTION

The task of change-point detection is of major importance in a number of scientific disciplines, ranging from engineering and sociology to economics and environmental studies.

The aim of the change-point identification task is to detect partitions in a sequence of observations, in order for the data in each block to be statistically "similar," in other words, to be distributed according to a common probability distribution.

Figure 1 shows the number of deadly accidents per year in the coal mines in England spanning the years 1851-1962. Looking at the graph, it is readily observed that the "front" part of the graph looks different from its "back" end, with a change around 1890-1900. As a matter of fact, in 1890, new health and safety regulations were introduced, following pressure from the coal miners' unions. We will use the poisson distribution.



Coal mining data

x=[4 5 4 1 0 4 3 4 0 6 3 3 4 0 2 6 3 3 5 4 5 3 1 4 4 1 5 5 3 4 2 5 2 2 ...
3 4 2 1 3 2 2 1 1 1 1 3 0 0 1 0 1 1 0 0 3 1 0 3 2 2 0 1 1 1 0 1 0 1 0 ...
0 0 2 1 0 0 0 1 1 0 2 3 3 1 1 2 1 1 1 1 2 4 2 0 0 0 1 4 0 0 0 1 0 0 0 ...
0 0 1 0 0 1 0 1]

Problem 2 submitted by the team

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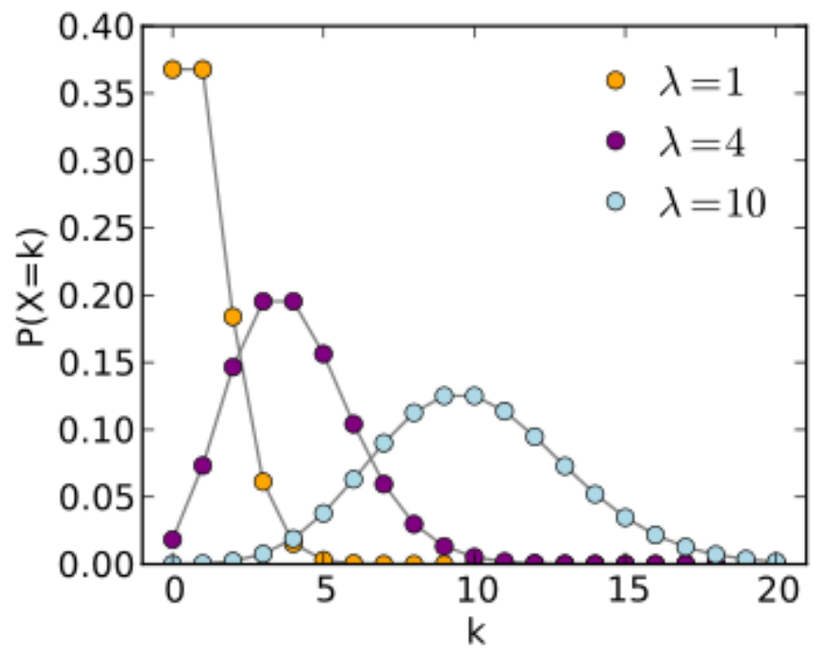
Let x_n be a discrete random variable that corresponds to the count of an event, for example, the number of requests for telephone calls within an interval of time, requests for individual documents on a web server, particle emissions in radioactive materials, number of accidents in a working environment, and so on. We adopt the Poisson process to model the distribution of x_n, that is,

$$P(x;\lambda) = \frac{(\lambda\tau)^x}{x!}e^{-\lambda\tau} \quad x=0,1,2,\ldots$$



Poisson processes have been widely used to model the number of events that take place in a time interval, τ. For our example, we have chosen $\tau = 1$. The parameter λ is known as the *intensity* of the process

Problem 2 submitted by the team

We assume that our observations, $x_n,\ n = 1, 2, \ldots, N$, have been generated by two different Poisson processes, $P(x; \lambda_1)$ and $P(x; \lambda_2)$. Also, the change of the model has taken place suddenly at an unknown time instant, n_0. Our goal is to estimate the posterior,

$$P(n_0|\lambda_1, \lambda_2, \boldsymbol{x}_{1:N}).$$

Moreover, the exact values of λ_1 and λ_2 are not known. The only available information is that the Poisson process intensities, $\lambda_i,\ i = 1, 2$, are distributed according to a (prior) gamma distribution, that is,

$$p(\lambda) = \text{Gamma}(\lambda|a, b) = \frac{1}{\Gamma(a)} b^a \lambda^{a-1} \exp(-b\lambda),$$

for some known positive values a, b. We will finally assume that we have no prior information on when the time of change occurred; thus, the prior is chosen to be the uniform distribution, $P(n_0) = \frac{1}{N}$. Based on the previous assumptions, the corresponding joint distribution is given by,

$$p(n_0, \lambda_1, \lambda_2, \boldsymbol{x}_{1:N}) = p(\boldsymbol{x}_{1:N}|\lambda_1, \lambda_2, n_0) p(\lambda_1) p(\lambda_2) P(n_0)$$

or

$$p(n_0, \lambda_1, \lambda_2, \boldsymbol{x}_{1:N}) = \prod_{n=1}^{n_0} P(x_n|\lambda_1) \prod_{n=n_0+1}^{N} P(x_n|\lambda_2) p(\lambda_1) p(\lambda_2) P(n_0).$$

Taking the logarithm in order to get rid of the products, and integrating out respective variables, the following conditionals needed in Gibbs sampling are obtained to prove

Problem 2 submitted by the team

2(A) prove the conditional probabilities to prepare for Gibbs sampling:

Taking the logarithm in order to get rid of the products, and integrating out respective variables, the following conditionals needed in Gibbs sampling are obtained

$$p(\lambda_1|n_0,\lambda_2,\boldsymbol{x}_{1:N}) = \text{Gamma}(\lambda_1|a_1,b_1),$$

with

$$a_1 = a + \sum_{n=1}^{n_0} x_n, \quad b_1 = b + n_0,$$

$$p(\lambda_2|n_0,\lambda_1,\boldsymbol{x}_{1:N}) = \text{Gamma}(\lambda_2|a_2,b_2),$$

$$a_2 = a + \sum_{n=n_0+1}^{N} x_n, \quad b_2 = b + (N - n_0),$$

and

$$\ln P(n_0|\lambda_1,\lambda_2,\boldsymbol{x}_{1:N}) \cong \ln\lambda_1 \sum_{n=1}^{n_0} x_n - n_0\lambda_1 + \ln\lambda_2 \sum_{n=n_0+1}^{N} x_n - (N - n_0)\lambda_2, \quad n_0 = 1,2,\ldots,N.$$

The last line for ln(P) just gives the log of the products of independent Poisson probabilities once the Poisson intensities $\lambda_{1,2}$ are determined from the Gamma distributions for a particular n_0. λ_1 up to year n_0 and λ_2 from year n_0+1 to year N. ~ indicates the normalization factor which has to be taken into account. See next page for how to draw Gibbs sampling from discrete probabilities.

For discrete probabilities P_i , with u ~ U(0,1) uniform random number in the (0,1) interval:

- Define $a_k = \sum_{i=1}^{k-1} P_i$, $b_k = \sum_{i=1}^{k} P_i$, $k = 1, 2, \ldots, K$, $a_1 = 0$.
- **For** $i = 1, 2, \ldots,$ **Do**
  - $u \sim \mathcal{U}(0, 1)$
  - Select

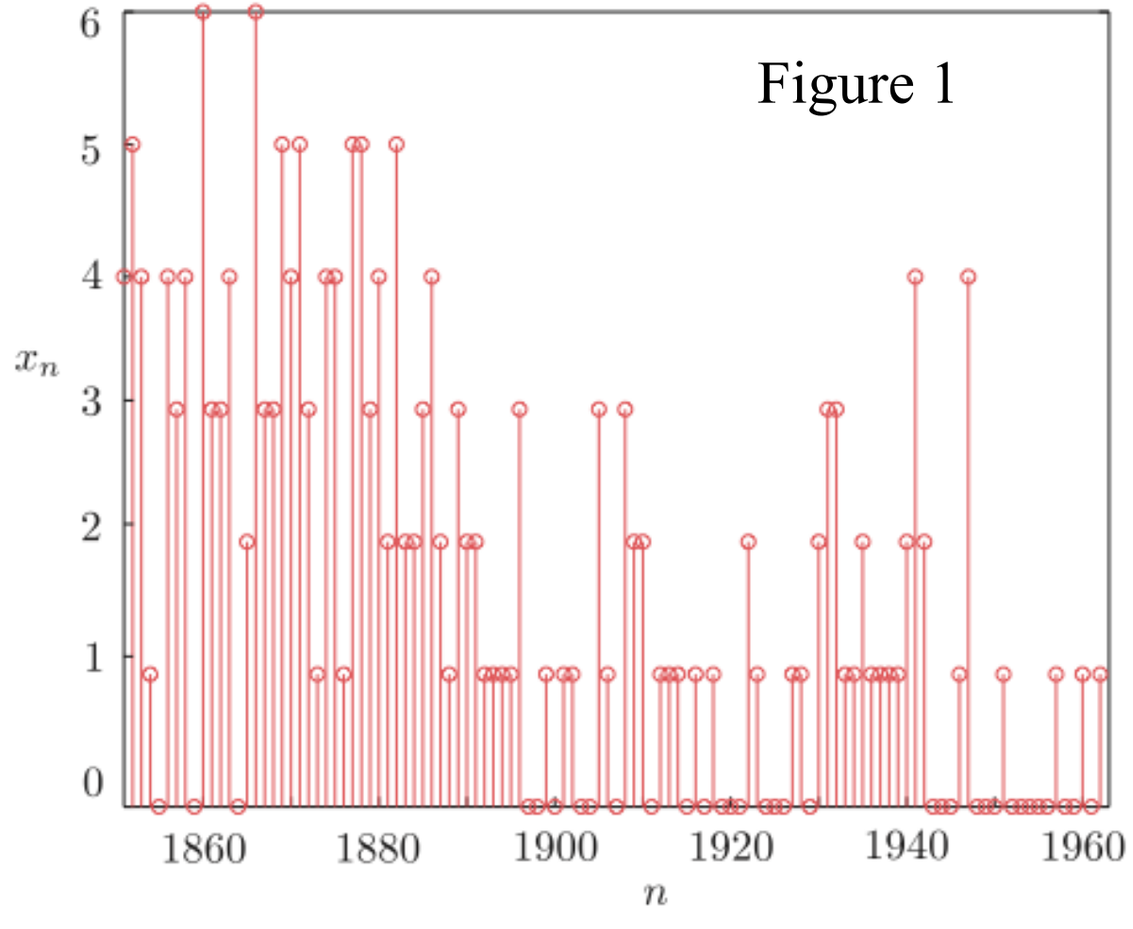
$$x_k \text{ if } u \in [a_k, b_k), \quad k = 1, 2, \ldots, K$$

- **End For**

Problem 2 submitted by the team

Figure 1 shows the number of deadly accidents per year in the coal mines in England spanning the years 1851-1962. Looking at the graph, it is readily observed that the "front" part of the graph looks different from its "back" end, with a change around 1890-1900. As a matter of fact, in 1890, new health and safety regulations were introduced, following pressure from the coal miners' unions. We will use the model explained before and draw samples in order to determine the point, n_0, where a change in the statistical distributions describing the data occurred

a and b were chosen equal to $a = 2$ and $b = 1$, although the obtained results are not sensitive



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0 0 2 1 0 0 0 1 1 0 2 3 3 1 1 2 1 1 1 1 2 4 2 0 0 0 1 4 0 0 0 1 0 0 0 ...
0 0 1 0 0 1 0 1]

Problem 2 submitted by the team

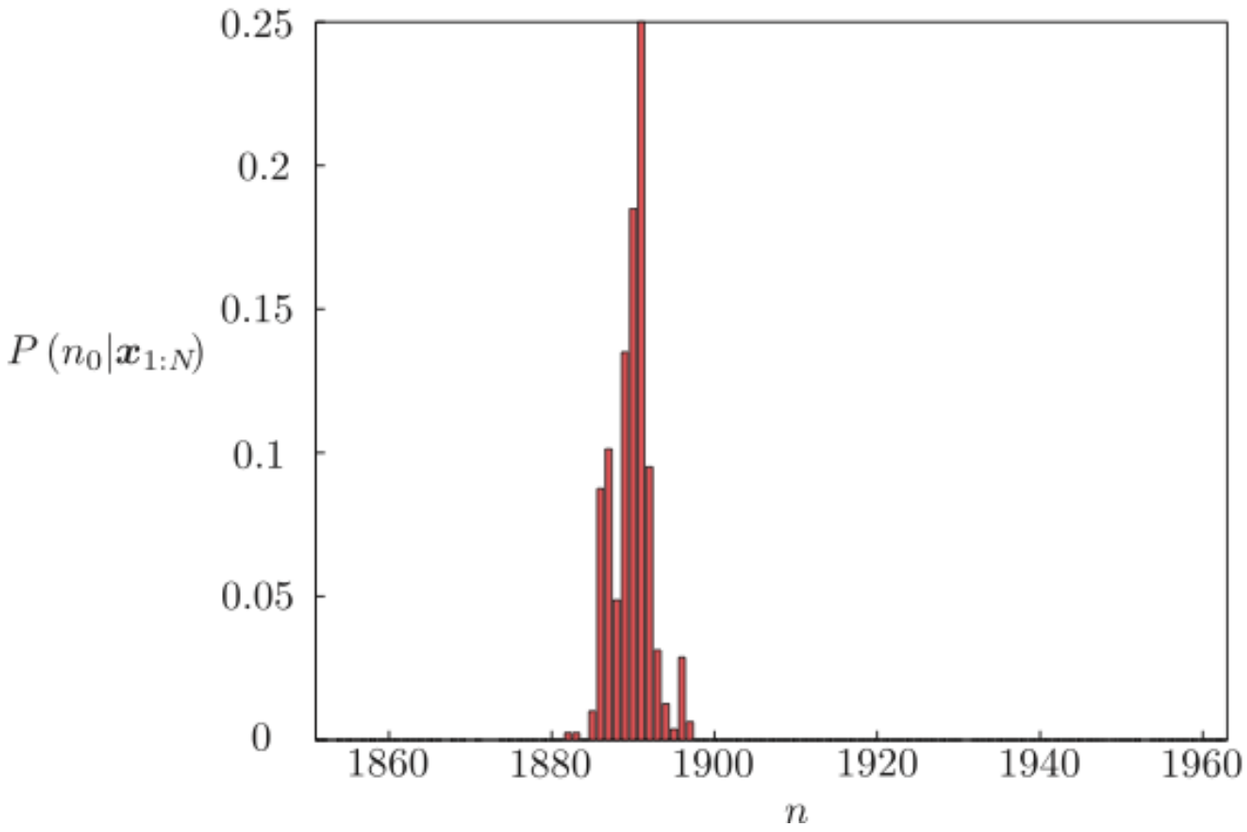
2(B) Implement the Gibbs sampling of the Markov Chain Monte Carlo:

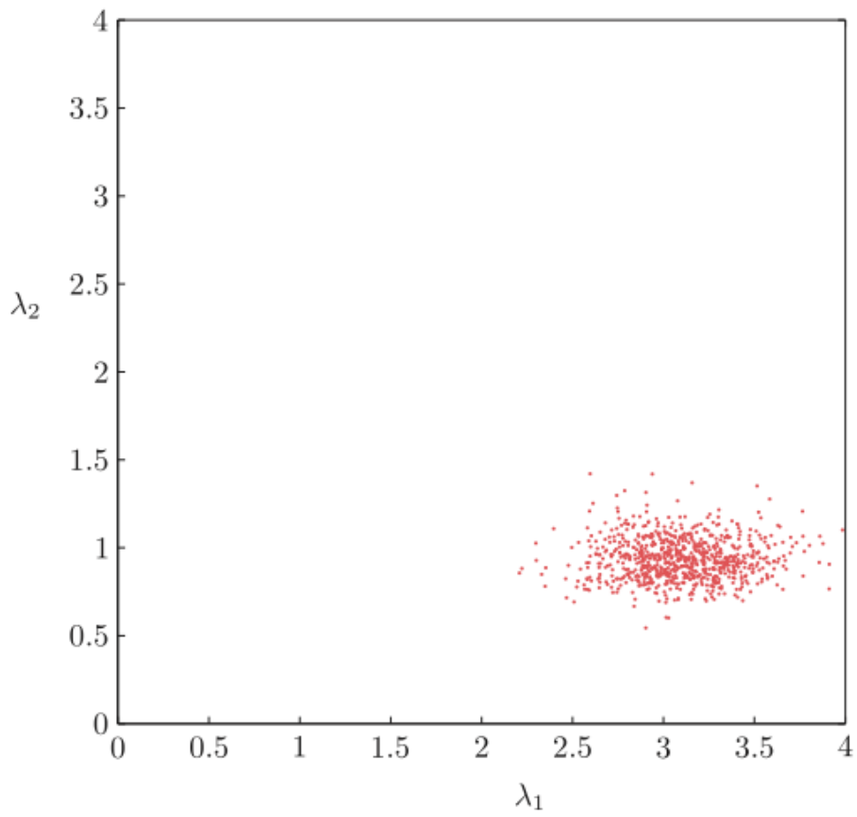
Gibbs sampling for change-point detection

- Having obtained $\boldsymbol{x}_{1:N} := \{x_1, \ldots, x_N\}$, select a and b.
- Initialize $n_0^{(0)}$
- **For** $i = 1, 2, \ldots,$ **Do**
  - $\lambda_1^{(i)} \sim \text{Gamma}\left(\lambda | a + \sum_{n=1}^{n_0^{(i-1)}} x_n,\ b + n_0^{(i-1)}\right)$
  - $\lambda_2^{(i)} \sim \text{Gamma}\left(\lambda | a + \sum_{n=n_0^{(i-1)}+1}^{N} x_n,\ b + (N - n_0^{(i-1)})\right)$
  - $n_0^{(i)} \sim P(n_0 | \lambda_1^{(i)}, \lambda_2^{(i)}, x_{1:N})$
- **End For**

Problem 2 submitted by the team

2(C) Plot the n_0 probably distribution and λ_1, λ_2 from Gibbs sampling:





2(D) What are the means of n_0, λ_1, λ_2 ?

2(E) Estimate the MC errors on n_0, λ_1, λ_2 from the independent MC configurations of the simulations

2(F) Estimate the MC errors on n_0, λ_1, λ_2 from the independent MC configurations of the simulations

Problem 2 (phys 239 only) submitted by the team

2(G) Compare your Gibbs sampling based simulation with Metropolis Monte Carlo